

Existence and behavior of steady solutions on an interval for general hyperbolic-parabolic systems of conservation laws

BENJAMIN MELINAND*AND KEVIN ZUMBRUN†

August 19, 2022

Keywords: Steady solutions, hyperbolic-parabolic conservation laws, Evans function.

Abstract

We study the inflow-outflow boundary value problem on an interval, the analog of the 1D shock tube problem for gas dynamics, for general systems of hyperbolic-parabolic conservation laws. In a first set of investigations, we study existence, uniqueness, and stability, showing in particular local existence, uniqueness, and stability of small amplitude solutions for general symmetrizable systems. In a second set of investigations, we investigate structure and behavior in the small- and large-viscosity limits.

A phenomenon of particular interest is the generic appearance of characteristic boundary layers in the inviscid limit, arising from noncharacteristic data for the viscous problem, even of arbitrarily small amplitude. This induces an interesting new type of “transcharacteristic” hyperbolic boundary condition governing the formal inviscid limit.

1 Introduction

In this paper, inspired by recent results in [MZ19, BMZ21] on the “1D shock tube problem” for gas dynamics, i.e., steady flow on a bounded interval, with noncharacteristic inflow/outflow boundary conditions, we here begin the systematic study of the “generalized 1D shock problem” for arbitrary systems of hyperbolic-parabolic conservation laws.

Our goals are two-fold: first, to add context and larger foundation to the somewhat special analyses of [MZ19, BMZ21], and, second, to introduce what seems to be a family of new and interesting types of hyperbolic and hyperbolic-parabolic problems for general conservation laws.

*CEREMADE, CNRS, Université Paris-Dauphine, Université PSL, 75016 PARIS, FRANCE; melinand@ceremade.dauphine.fr.

†Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grant no. DMS-1700279.

1.1 Equations and assumptions

Following [SZ01, Zum10], we consider steady solutions of general viscous conservation laws:

$$(1.1) \quad \partial_t f^0(U) + f(U)_x = (B(U)U_x)_x, \quad 0 < x < 1, \quad t > 0,$$

together with the boundary conditions

$$(1.2) \quad U(0) = U_0 = \begin{pmatrix} U_{0I} \\ U_{0II} \end{pmatrix} \quad \text{and} \quad U_{II}(1) = U_{1II}.$$

In this paper we assume that $U \in \mathcal{U} \subset \mathbb{R}^n$,

$$U = \begin{pmatrix} U_I \\ U_{II} \end{pmatrix}, \quad f^0 = \begin{pmatrix} f_I^0 \\ f_{II}^0 \end{pmatrix}, \quad f = \begin{pmatrix} f_I \\ f_{II} \end{pmatrix} \in \mathbb{R}^r \times \mathbb{R}^{n-r}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix}, \quad B_{22} \in M_{n-r}(\mathbb{R}),$$

and df^0 is invertible and lower block triangular, without loss of generality

$$df^0(U) = \begin{pmatrix} \text{Id}_r & 0 \\ (df_{II}^0)_I & (df_{II}^0)_{II} \end{pmatrix}$$

with $\det((df_{II}^0)_{II}) > 0$. We also make the following assumptions:

(H0) f and B are smooth.

(H1) The eigenvalues of $(df_{II}^0)_{II}^{-1}(U)B_{22}(U) + ((df_{II}^0)_{II}^{-1}(U)B_{22}(U))^T$ are positive for any U .

(H2) The eigenvalues of $(df_I)_I(U)$ are real and positive for any U .

Condition **(H1)** corresponds to strict parabolicity of (1.1) with respect to U_{II} , consistent with Dirichlet boundary conditions on $U_{II}(0), U_{II}(1)$. Note that $B_{22}(U)$ is necessary invertible and $\det B_{22}(U) > 0$ for any U (since $\det((df_{II}^0)_{II}) > 0$). The first part of Condition **(H2)** corresponds to hyperbolicity of (1.1) with respect to variable U_I . The second part of Condition **(H2)** means that the flow moves from the left to the right, so that the hyperbolic part of the flow, U_I , is completely entering the domain at the left boundary, consistent with Dirichlet boundary conditions on $U_I(0)$.

In the following, we denote $A^0(U) = df^0(U)$, $A(U) = df(U)$, with

$$A^0 = \begin{pmatrix} \text{Id}_r & 0 \\ A_{21}^0 & A_{22}^0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} \in M_r(\mathbb{R}).$$

1.2 Main results

In Section 2, we categorize essentially completely existence, uniqueness and stability of small-amplitude steady solutions for general viscous conservation laws (1.1) under steadily increasing assumptions, culminating with the physically natural assumption of existence

of a convex entropy. We discuss afterward in a more partial and speculative way the issues of global existence, uniqueness, and stability, that is, the situation for *large-amplitude solutions*. In particular, we outline a strategy based on entropy dissipation and Brouwer degree for global uniqueness of constant solutions and global existence of large-amplitude systems, generalizing and illuminating the one introduced in [BMZ21] in the context of gas dynamics.

In Section 3, we give a complete study of structure and behavior of steady solutions in the scalar case in the small viscosity limit.

In Section 4, we turn to the study of structure and behavior of steady solutions, in the small- and large-viscosity limits. Even for the simple case of isentropic gas dynamics, there is a rich “zoo” of possible solution patterns in the inviscid limit similar to what we observe in Section 3, featuring standing shocks, left and right boundary layers, and, the most novel feature, a new type of characteristic boundary layer appearing not as a boundary case, but *generically* in “rarefying,” or “expansive” solutions. These induce in the formal inviscid limit a new type of “transcharacteristic” boundary condition accomodating perturbations on either side of a characteristic state, hence requiring different numbers of boundary conditions; see Section 4.1.3. This phenomenon occurs for *any* rarefying steady solution, even of arbitrarily small amplitude, hence must be dealt with in order to produce a hyperbolic description of dynamics.

1.3 Discussion and open problems

A very interesting finding of [BMZ21] is that nonuniqueness may hold for systems with convex entropy, in particular even for gas dynamics with an artificially devised equation of state. Yet, it appears (numerically) to hold for the standard polytropic equation of state. A very interesting open problem is to verify this analytically. Likewise, the proof or disproof of existence for more general equations of state, or more general equations such as magnetohydrodynamics (MHD) are important open problems. Existence of large-amplitude multi-D solutions is a challenging more long term goal, even for gas dynamics, that appears to require further ideas for its resolutions.

As regards structure and asymptotic behavior, a very interesting open problem is to determine the possible feasible hyperbolic configurations for general systems, similarly as done here for isentropic gas dynamics. The understanding of hyperbolic behavior for the limiting zero-viscosity dynamics is another challenging direction. The handling of characteristic boundary layers, and resulting transcharacteristic type boundary conditions is a particularly intriguing piece of this puzzle. Similarly, the understanding of large-viscosity behavior for general systems is another interesting direction for study.

The most intriguing open problem is one that has not been addressed at all here, namely structure of the corresponding *multi-D solutions* in the inviscid limit. This could be thought of as a combination of compressible Pouseille flow and the 1-D shock problem; the results should be very interesting indeed. In particular, even for small data, for which multi-D existence and uniqueness are known [KK97], the question of structure seems not to have yet been addressed. An especially interesting question seems to be what is the role in multi-d of

the characteristic boundary layer configurations we have seen here in 1-D. Presumably such configurations are there in the construction of [KK97], but their numerical and asymptotic description is still wanting.

Acknowledgement. We thank Blake Barker for many helpful discussions during the closely related project [BMZ21], and for his generous aid both numerical and analytical.

2 Existence, uniqueness and spectral stability

2.1 The linear case

We assume in this part that df and B are both constant. We have the following proposition.

Proposition 2.1. *If df and B are both constant and conditions **(H0)**-**(H2)** are satisfied, Problem (1.1) has a unique steady state that satisfies the boundary condition (1.2) if and only if*

$$(2.1) \quad \sigma(B_{22}^{-1}(A_{22} - A_{21}A_{11}^{-1}A_{12})) \cap 2i\pi\mathbb{Z} \setminus \{0\} = \emptyset.$$

Remark 2.1. *Condition (2.1) is a compatibility condition between the parabolic and the hyperbolic part. A similar condition was assumed for the study of quasilinear noncharacteristic boundary layers (on the half-line) in, for instance, [Mét06, Lemma 5.1.3] or [Mét04]. For example, the following system does not satisfy Condition (2.1)*

$$U_t + \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix} U_x = U_{xx}, \quad 0 < x < 1,$$

and any constant state $\hat{U}$ must satisfy $\hat{U}(0) = \hat{U}(1)$.

Proof of Proposition 2.1. We can rewrite the problem as

$$\begin{cases} A_{11}U'_I + A_{12}U'_{II} = 0, \\ A_{21}U'_I + A_{22}U'_{II} = B_{22}U''_{II}. \end{cases}$$

Then, integrating, we get

$$\begin{cases} A_{11}U_I + A_{12}U_{II} = C_1, \\ A_{21}U_I + A_{22}U_{II} + C_2 = B_{22}U'_{II} \end{cases}$$

where $C_1 = A_{11}U_{I0} + A_{12}U_{II0}$ and C_2 is a constant that has to be determined. Then, since A_{11} is invertible, we obtain

$$(2.2) \quad \begin{cases} U'_I = A_{11}^{-1}C_1 - A_{11}^{-1}A_{12}U_{II}, \\ U'_{II} = B_{22}^{-1}(A_{22} - A_{21}A_{11}^{-1}A_{12})U_{II} + B_{22}^{-1}(C_2 + A_{21}A_{11}^{-1}C_1). \end{cases}$$

Denoting $\tilde{A} = B_{22}^{-1}(A_{22} - A_{21}A_{11}^{-1}A_{12})$ and $\tilde{C} = B_{22}^{-1}(C_2 + A_{21}A_{11}^{-1}C_1)$, we solve

$$\begin{cases} U'_{II} = \tilde{A}U_{II} + \tilde{C} \\ U_{II}(0) = U_{II0}. \end{cases}$$

We decompose $\tilde{A}$ as $\tilde{A} = P^{-1} \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} P$ where P and F_2 are invertible and F_1 is strictly upper triangular. For example, $\text{blockdiag}\{F_1, F_2\}$ could be taken to be a Jordan form for $\tilde{A}$, with F_1 the 0-generalized eigenspace part. We get that

$$U_{II}(1) = e^{\tilde{A}} U_{II}(0) + P^{-1} \begin{pmatrix} \int_0^1 e^{sF_1} ds & 0 \\ 0 & F_2^{-1} (e^{F_2} - I) \end{pmatrix} P \tilde{C}.$$

Thus, we see that the map $C_2 \rightarrow U_{II}(1)$ is invertible if and only if $\sigma(\tilde{A}) \cap 2i\pi\mathbb{Z} \setminus \{0\} = \emptyset$. $\square$

2.2 Almost-constant steady states

We next study the existence of steady states for system (1.1)-(1.2) for data lying near a constant state satisfying (2.1). We seek solutions $\hat{U}$ of

$$(2.3) \quad (f(\hat{U}))_x = \left(B(\hat{U}) \hat{U}_x \right)_x, \quad 0 < x < 1, \quad \hat{U}(0) = U_0, \quad \hat{U}_{II}(1) = U_{1II}.$$

Similarly as done for gas dynamics in [MZ19, BMZ21], for U_0 and U_{1II} fixed, we define the map

$$\Psi : C_2 \in \mathbb{R}^{n-r} \rightarrow U_{II}(1) - U_{1II} \in \mathbb{R}^{n-r}$$

such that the maximal solution U of the ODE

$$(2.4) \quad B_{22}(U) U'_{II} = f_{II}(U) - f_{II}(U_0) + B_{22}(U_0) C_2, \quad U_{II}(0) = U_{0II}, \quad f_I(U) = f_I(U_0)$$

is defined on $[0, 1]$. Thus, profiles are equivalent to roots C_2 of Ψ .

Proposition 2.2. *Let $U_0 \in \mathbb{R}^n$ and assume conditions **(H0)**-**(H2)** are satisfied. Assume that Condition (2.1) is satisfied for $A = df(U_0)$ and $B = B(U_0)$. There exists $\delta > 0$ and $\varepsilon > 0$ such that for any U_{1II} with $|U_{0II} - U_{1II}| \leq \delta$, there exists a unique solution $\hat{U}$ of (2.3) satisfying*

$$|\hat{U}'_{II}(0)| \leq \varepsilon.$$

Moreover, the solution is nondegenerate: i.e., corresponds to a nondegenerate root of Ψ .

Remark 2.3. *We do not claim that for U_{1II} close enough to U_{0II} , there exists a unique solution of (2.3). The previous theorem only gives local uniqueness, even for small data.*

Proof of Proposition 2.2. Let us fix $U_0 \in \mathbb{R}^n$, taking without loss of generality $f_I(U_0) = 0$. First, we notice that for $U_{1II} = U_{0II}$, $\hat{U} \equiv U_0$ is a solution of (2.3). Then, by continuous dependence on parameters on the ODE, **(H2)** and the implicit function theorem on the constraint $f_I(U_I, U_{II}) = 0$, for (U_{1II}, C_2) close enough to $(U_{0II}, 0)$, one can express U_I as a function of U_{II} and the maximal solution U of (2.4) is defined on $[0, 1]$. Therefore, we can define the map Ψ on a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^{n-r}$. The function Ψ is $\mathcal{C}^1$ on this domain

and $\Psi(0) = 0$. Then, for any $D \in \mathbb{R}^{n-r}$, $d\Psi(0) \cdot D = V_{II}(1)$, with V_{II} solving the variational equation

$$B_{22}(U_0)V'_{II} = (A_{22} - A_{21}A_{11}^{-1}A_{12})(U_0)V_{II} + C_2 + (A_{21}A_{11}^{-1})(U_0)C_1, \quad V_{II}(0) = 0,$$

with $C_1 = A_{11}(U_0)U_{I0} + A_{12}(U_0)U_{II0}$. As in the proof of Proposition 2.1, we can solve this ODE and, using Condition (2.1), we obtain that $d\Psi(0)$ is invertible. The result then follows from the implicit function theorem, as does nondegeneracy. $\square$

2.2.1 The Evans function

Linearizing (1.1) around a steady solution $\hat{U}$, we obtain eigenvalue equations

$$(2.5) \quad \lambda A^0 V + (AV)_x = (BV_x)_x, \quad 0 < x < 1, \quad t > 0,$$

A^0 , A , B depending on x , with homogeneous boundary conditions $U(0) = 0$, $U_{II}(1) = 0$. In particular, $A_{11}^0 = I_r$, $A_{12}^0 = 0$, $A_{11} = df_I(\hat{U})$ is invertible, $A_{12} = df_{II}(\hat{U})$, and $B = B(\hat{U})$ with B_{22} invertible. Thus, denoting d/dx as $'$, we may rewrite (2.5) as a first-order system

$$(2.6) \quad \begin{pmatrix} A_{11}V_I \\ V_{II} \\ B_{22}V'_{II} \end{pmatrix}' = \begin{pmatrix} -\lambda A_{11}^{-1} & -A'_{12} & -A_{12}B_{22}^{-1} \\ 0 & 0 & B_{22}^{-1} \\ \lambda(A_{21}^0 - A_{21}A_{11}^{-1})A_{11}^{-1} + \alpha & \lambda A_{22}^0 + \beta & \gamma \end{pmatrix} \begin{pmatrix} A_{11}V_I \\ V_{II} \\ B_{22}V'_{II} \end{pmatrix}$$

with

$$\alpha = A'_{21}A_{11}^{-1} + A_{21}(A_{11}^{-1})', \beta = A'_{22} - A_{21}A_{11}^{-1}A'_{12}, \gamma = (A_{22} - A_{21}A_{11}^{-1}A_{12})B_{22}^{-1}.$$

The *Evans function* may thus be defined via a “shooting” construction, similarly as in [AGJ90, GZ98] for the whole line or [Rou01, SZ01] for the half-line case, as

$$(2.7) \quad D(\lambda) := \det(\mathcal{Z}_1, \dots, \mathcal{Z}_{2n-r})|_{x=1},$$

where $\mathcal{Z}_1, \dots, \mathcal{Z}_{n-r}$ are solutions with data $\mathcal{Z}_j(0) = (0, 0, B_{22}e_j^{n-r})$ prescribed at $x = 0$ and $\mathcal{Z}_{n-r+1}, \dots, \mathcal{Z}_n$ and $\mathcal{Z}_{n+1}, \dots, \mathcal{Z}_{2n-r}$ are solutions with data

$$\mathcal{Z}_{n-r+j}(1) = (A_{11}e_j^r, 0, 0), \quad j = 1, \dots, r$$

and $\mathcal{Z}_{n+j}(1) = (0, 0, B_{22}e_j^{n-r})$, $j = 1, \dots, n-r$ prescribed at $x = 1$ and where e_j^p is the j -th element of the standard basis of $\mathbb{R}^p$. Evidently, $D(\lambda)$ vanishes if and only if there exists a solution to the eigenvalue equation, hence *spectral stability is equivalent to nonvanishing of the Evans function on $\Re \lambda \geq 0$* .

2.2.2 Stability index, uniqueness, and Zumbrun-Serre/Rousset lemma

Clearly D is real-valued for real λ . It is readily seen (see, e.g., [SZ01, Zum10] in the half-line case) that $D(\lambda) \neq 0$ for λ real and sufficiently large, hence we may define as in [GZ98] the *Stability index*

$$(2.8) \quad \mu := \operatorname{sgn} D(0) \left(\lim_{\lambda \rightarrow +\infty_{\text{real}}} \operatorname{sgn} D(\lambda) \right).$$

The index μ determines the parity of the number of roots of the Evans function with positive real part, with $+1$ corresponding to “even” and -1 to “odd”. Thus, $\mu = +1$ is a *necessary condition for stability*. The following result analogous to the Zumbrun-Serre/Rousset lemmas of [ZS99, Rou01] in the whole- and half-line case, relates low-frequency stability $D(0) \neq 0$ and stability index information to transversality of the steady profile solution of the standing-wave ODE (cf. [BMZ21, Lemma 6.1] for gas dynamics).

Lemma 2.4. *The zero-frequency limit $D(0)$ is equal to $\det B_{2,2}$ multiplied by the Jacobian determinant $\det(d\Psi(C_2))$ associated with problem (2.4) evaluated at any root C_2 ; in particular,*

$$(2.9) \quad \operatorname{sgn} D(0) = \operatorname{sgn} \det d\Psi(C_2).$$

Proof. This follows from the observation that for $\lambda = 0$ the eigenvalue equation reduces to the variational equation for (2.4) for the steady profile, with $C_1 = 0$ and $C_2 = B_{22}U'_{II}(0)$ imposed by the homogeneous boundary conditions at $x = 0$. Noting that $d\Psi(C_2)$ is equal to the value of the solution $U_{II}(1)$ of (2.2) with matrix-valued data $\tilde{C}_2 = C_2 = \operatorname{Id}_{n-r}$, whereas $D(0)$ is equal to the determinant of the solution with matrix-valued data $U'_{II}(0) = B_{22}\operatorname{Id}_{n-r}$, we have $D(0) = \det B_{22} \det d\Psi(C_2)$, giving the result by $\det B_{22} > 0$. $\square$

2.3 Symmetrizable systems

The spectral condition (2.1) is satisfied for many physical systems, in particular ones that are *symmetrizable* in the sense of [KS88]. We first have the following technical lemma.

Lemma 2.5. *If $\tilde{A}$ is symmetric and $\tilde{B} + \tilde{B}^T > 0$, then $\sigma(\tilde{B}^{-1}\tilde{A}) \cap i\mathbb{R} \subset \{0\}$.*

Proof. If $\tilde{B}v = i\tau\tilde{A}v$ for $\tau \neq 0$, we get

$$2i\tau\langle v, \tilde{A}v \rangle = \langle v, (\tilde{B} + \tilde{B}^T)v \rangle + \langle v, (\tilde{B} - \tilde{B}^T)v \rangle.$$

and since $\tilde{B} + \tilde{B}^T > 0$, $v = 0$. $\square$

We recall that (1.1) is said to be symmetrizable if:

(H3) There exists a smooth map $S : U \in \mathbb{R}^n \mapsto S(U)$ such that, for any $U \in \mathbb{R}^n$, $S(U)A^0(U)$ is symmetric positive definite, $S(U) = \begin{pmatrix} S_{11}(U) & 0 \\ 0 & S_{22}(U) \end{pmatrix}$, $S(U)A(U)$ is symmetric and $S_{22}(U)B_{22}(U) + (S_{22}(U)B_{22}(U))^T > 0$.

Then we have the following useful lemma.

Lemma 2.6. *Under assumption (H3), $\sigma(B_{22}^{-1}(A_{22} - A_{21}A_{11}^{-1}A_{12})) \cap i\mathbb{R} \subset \{0\}$.*

Proof. We note first that by assumption $S_{22}A_{21} = (S_{11}A_{12})^T$. Then we write

$$B_{22}^{-1}(A_{22} - A_{21}A_{11}^{-1}A_{12}) = (S_{22}B_{22})^{-1}(S_{22}A_{22} - (S_{11}A_{12})^T(S_{11}A_{11})^{-1}S_{11}A_{12})$$

and the result follows from the previous lemma. $\square$

Theorem 2.7. *For symmetrizable systems under Conditions (H0)-(H3), almost-constant solutions of almost constant data exist and are locally unique, nondegenerate, and spectrally stable.*

Proof. The existence, local uniqueness, and nondegeneracy are immediate consequences of Proposition 2.2 and Lemma 2.6. Furthermore, one can easily adapt [MZ19, Prop. 3.2] and prove that the spectrum of the linearized operator about a steady state $\hat{U}$ only contains eigenvalues. We then consider the eigenvalue problem

$$\lambda A^0(\hat{U})V + (A(\hat{U})V)_x = \left(B(\hat{U})V_x + dB(\hat{U})(V)\hat{U}_x \right)_x, \quad V(0) = 0, \quad V_{II}(1) = 0.$$

If $\hat{U} \equiv U_0$, one can check that

$$\begin{aligned} \Re(\lambda) (S(U_0)A^0(U_0)V, V)_{L^2(0,1)} + (S_{22}(U_0)B_{22}(U_0)V_{IIx}, V_{IIx})_{L^2(0,1)} \\ + \frac{1}{2} |\sqrt{S_{11}(U_0)A_{11}(U_0)}V_I(1)|^2 = 0. \end{aligned}$$

Noting, since $A_{11}(U)$ is symmetric for the inner product associated to $S_{11}(U)$, that Conditions (H2)-(H3) give $S_{11}(U_0)A_{11}(U_0) > 0$, we thus have that U_0 is spectrally stable.

Then, for almost-constant steady states (meaning $\hat{U}_x$ small enough), we use an appropriate Goodman-type estimate. We note that there exists a constant $C > 0$, such that for any $\alpha \geq 0$ and weight $\varphi(x) = e^{-\alpha x}$, we have

$$\begin{aligned} \Re(\lambda) \left(\varphi S(\hat{U})A^0(\hat{U})V, V \right)_{L^2} + \frac{1}{2} \varphi(1) \left| \sqrt{S_{11}(\hat{U})A_{11}(\hat{U})}V_I(1) \right|^2 \\ \leq -\frac{\alpha}{2} \left(\varphi S(\hat{U})A(\hat{U})V, V \right)_{L^2} + C|\hat{U}|_{L^\infty} (\varphi V, V)_{L^2} + \frac{\alpha^2}{2} \left(\varphi S(\hat{U})B(\hat{U})V, V \right)_{L^2} \\ - \left(\varphi S(\hat{U})B(\hat{U})V_x, V_x \right)_{L^2} + C|\hat{U}|_{L^\infty} (1 + \alpha) |\sqrt{\varphi}V_{IIx}|_{L^2} |\sqrt{\varphi}V_{II}|_{L^2}. \end{aligned}$$

Then, since the system is symmetrizable, there exists some constants $a, b > 0$ such that

$$\left[-\frac{\alpha}{2} \varphi S(\hat{U})A(\hat{U}) + C|\hat{U}|_{L^\infty} \varphi I_n + \frac{\alpha^2}{2} \varphi S(\hat{U})B(\hat{U}) \right] \leq \left(-\frac{a\alpha}{2} + C|\hat{U}|_{L^\infty} + \frac{b\alpha^2}{2} \right) \varphi I_n.$$

We choose $\alpha > 0$ so that $-\frac{a\alpha}{2} + C|\hat{U}|_{L^\infty} + \frac{b\alpha^2}{2} = -\frac{a\alpha}{4}$ (which is always possible for $|\hat{U}|_{L^\infty}$ small enough). Thus we get

$$\begin{aligned} \Re(\lambda) \left(\varphi S(\hat{U}) A^0(\hat{U}) V, V \right)_{L^2} &\leq -\frac{a\alpha}{4} |\sqrt{\varphi} V|_{L^2}^2 - \left(\varphi S(\hat{U}) B(\hat{U}) V_x, V_x \right)_{L^2} \\ &\quad + C|\hat{U}|_{L^\infty} (1 + \alpha) |\sqrt{\varphi} V_{Ix}|_{L^2} |\sqrt{\varphi} V_{II}|_{L^2}. \end{aligned}$$

Using Young's inequality together with the fact that $|\hat{U}|_{L^\infty}$ is small enough, we obtain

$$\Re(\lambda) \left(\varphi S(\hat{U}) A^0(\hat{U}) V, V \right)_{L^2} \leq -\frac{a\alpha}{8} |\sqrt{\varphi} V|_{L^2}^2.$$

Thus, we may again conclude that $\hat{U}$ is spectrally stable. See [MZ19, Section 3] for similar computations in the isentropic case. Alternatively, one could conclude by continuous dependence on coefficients of solutions of the eigenvalue ODE, as in [Zum10, GMMZ09]. $\square$

Remark 2.8. *Note that, contrary to the whole line situation (see for instance [Kaw83, KS88, Zum04]), we do not assume Kawashima's genuine coupling condition [Kaw83] in Theorem 2.7. The main reason behind this is that a steady state $\hat{U}$ of a purely hyperbolic system ($B \equiv 0$) on an interval under assumptions **(H0)**, **(H2)**, **(H3)** is stable, as in this case $f = f_I$ and $f_I(U_0) = f_I(\hat{U})$ enforces $\hat{U}$ constant, and also all characteristics move from left to right, sweeping perturbations out of the domain, with full Dirichlet conditions at the left boundary. Even better, any solution of Problem (1.1)-(1.2) initially close enough to $\hat{U}$ is equal to $\hat{U}$ after a finite time. Thus, the hyperbolic part of the system need not be coupled to the parabolic part in order to achieve stability, but is stable even by itself.*

Remark 2.9. *Using the same kind of energy estimates, one can also prove the nonlinear stability of an almost-constant steady state. See [MZ19, Section 6] for similar considerations in the isentropic gas dynamic case.*

Combining the argument of 2.1 with the results of Theorem 2.7, we may deduce not only nonvanishing of $D(0)/\mu$, $d\Psi$ but useful sign information, included here for definiteness.

Proposition 2.10. *For arbitrary solutions of symmetrizable systems,*

$$(2.10) \quad \mu = \operatorname{sgn} D(0) = \operatorname{sgn} \det d\Psi(C_2).$$

For constant solutions of symmetrizable systems, corresponding to $C_2 = C_2^$,*

$$(2.11) \quad \mu = \operatorname{sgn} \det d\Psi(C_2^*) = +1.$$

Proof. From the argument of Proposition 2.1, we find that the profile map $\Psi : C_2 \rightarrow U_{II}(1)$ satisfies $\det d\Psi(C_2^*) = \det \int_0^1 e^{\tilde{A}(1-s)} ds \neq 0$ at the value C_2^* corresponding to a constant solution, where $\tilde{A} = B_{22}^{-1}(A_{22} - A_{21}A_{11}^{-1}A_{12})$, and thus is invariant under homotopy within the class of symmetrizable systems, hence may take B_{22} without loss of generality to be

a multiple of the identity, hence, by symmetrizability, $\tilde{A}$ has real semisimple eigenvalues $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{n-r}$. Diagonalizing, we have by direct computation that $\text{sgn det } d\Psi(C_2^*) = \text{sgn det } \int_0^1 e^{\tilde{A}(1-s)} ds = \prod_{j=1}^{n-r} \text{sgn } \int_0^1 e^{\alpha_j(1-s)} ds = 1$, hence $\text{sgn det } d\Psi(C_2^*) = \text{sgn } D(0) = +1$.

Using spectral stability, $D(\lambda) \neq 0$ together with real-valuedness of the Evans function when restricted to the real axis, we may extend by a further homotopy in λ to conclude that the limit of $\text{sgn } D(\lambda)$ as $\lambda \rightarrow \infty$, real, is also $+1$. Summarizing, we have (2.11). By standard arguments [SZ01, Zum10], either WKB-type or using energy estimates, one has for general solutions of symmetrizable systems that $D(\lambda) \neq 0$ for λ real and large, hence $\lim_{\lambda \rightarrow +\infty} \text{sgn } D(\lambda) = +1$, giving (2.10), by homotopy in C_2 . $\square$

2.4 Systems with convex entropy

A system (1.1) is said to have a convex entropy [Kaw83, KS88] if it has an entropy/entropy flux pair $(\eta, q)(f^0) : \mathbb{R}^n \rightarrow \mathbb{R}^2$ such that

$$(2.12) \quad \frac{d^2 \eta}{(df^0)^2} > 0, \quad (d\eta/df^0)(df/df^0) = dq/df^0,$$

and

$$(2.13) \quad (df^0)^T \frac{d^2 \eta}{(df^0)^2} B + \left((df^0)^T \frac{d^2 \eta}{(df^0)^2} B \right)^t \geq 0,$$

with equality only on $\ker B$. It is a theorem of [KS88] that existence of a convex entropy implies symmetrizability, i.e., reducibility by coordinate change to a system satisfying **(H3)**. Thus, we may deduce local uniqueness information for systems with a convex entropy already by reference to Theorem 2.7.

2.4.1 Global uniqueness for constant data

Arguing directly, we may obtain under a mild additional assumption, much more. Namely, assume as holds for most physical systems that

$$(2.14) \quad (df_I)_I \text{ symmetric (hence positive definite).}$$

For gas dynamics, magnetohydrodynamics, and elasticity, $(df_I)_I = u \text{Id}_r$ and so (2.14) is trivially satisfied. By $\langle h, f_I(W + h, U_{II}) - f_I(W) \rangle = \langle h, \left(\int (df_I)_I(W + \theta h) d\theta \right) h \rangle > 0$, this yields the global solvability property

$$(2.15) \quad \text{For fixed } U_{II}, (df_I)(\cdot, U_{II}) \text{ is (globally) one-to-one.}$$

Alternatively, we may take $(df_I)_I + (df_I)_I^t > 0$, or just impose (2.15) without reference to (2.14). Then, we have the following *global* uniqueness result, for constant data only.

Theorem 2.11. *For systems (1.1)-(1.2) with a global convex entropy and satisfying **(H0)**-**(H3)** and (2.15), solutions of (2.3) for constant data $U_{0II} = U_{1II}$ are globally unique, nondegenerate (full rank), and spectrally stable, consisting exclusively of constant states.*

Proof. Following [Lax73], we obtain multiplying (1.1) by $d\eta/df^0$ and using (2.12)(ii) the equation

$$\eta_t + q_x = (d\eta/df^0)(BU_x)_x = ((d\eta/df^0)BU_x)_x - \langle df_0 U_x, \frac{d^2\eta}{(df^0)^2} U_x \rangle.$$

By (2.13), we have $\langle df_0 U_x, \frac{d^2\eta}{(df^0)^2} BU_x \rangle \geq 0$ with equality if and only if $BU_x = 0$. Thus, integrating the steady equation from $x = 0$ to $x = 1$, we obtain

$$(2.16) \quad (q(U) - d\eta(U)B(U)U')|_0^1 \leq 0,$$

with equality if and only if $BU' \equiv 0$ or equivalently $U'_{II} \equiv 0$.

On the other hand, integrating the U_I equation, we have $f_I(U) \equiv \text{constant}$, whence, by (H2) and (2.15), $U_I(0) = U_I(1)$, and so $U(0) = U(1)$. Thus, $q(U)|_0^1$ vanishes in (2.16). At the same time, by addition of an arbitrary linear function, we may take η without loss of generality to satisfy $d\eta(U(0)) = d\eta(U(1)) = 0$, whence the entire lefthand side of (2.16) vanishes. Then, equality holds in (2.16) and so we must have $U'_{II} \equiv 0$ and therefore $U_{II} \equiv U_{II}(0)$ and $f_I \equiv f_I(U(0))$. Applying (H2) and the implicit function theorem, we find that we may solve for a unique value of U_I in a neighborhood of $U_I(0)$ as a function of the constants U_{II} and $f_I(U)$. Since $U_I(x)$ varies continuously starting at $U_I(0)$, it can thus never escape this local neighborhood, and so $U_I(x) \equiv U_I(0)$ as well. This gives global uniqueness of $U \equiv U(0)$. Nondegeneracy and spectral stability follow by Theorem 2.7. $\square$

Remark 2.12. *The assumption (2.14) seems possibly related to the circle of ideas around entropy and symmetrizability. It would be very interesting to identify sharp criteria for (2.14) assuming existence of a complex entropy; however, we have not found such.*

2.4.2 Global existence for general data

Following [BMZ21], define the feasible set $\mathcal{C}$ as the connected component of the set of parameters C_2 corresponding to constant solutions of the open set of C_2 for which the solution of (2.4) is defined on $[0, 1]$ and remains in the interior of its domain of definition $\mathcal{U}$. If Ψ is “proper” in the sense that the inverse image in $\mathcal{U}$ of a compact set in $\mathcal{C}$ is compact, then we may conclude general existence from the special uniqueness result of Theorem 2.11.

Corollary 2.13. *For systems possessing a convex entropy for which Ψ is proper in the above sense with respect to $\mathcal{U}$, $\mathcal{C}$, there exists a steady profile solution for any data $U(0), U_2(1)$.*

Proof. The map Ψ is continuous on $\mathcal{C}$, by continuous dependence of solutions of ODE, For Ψ proper in the above sense, one may thus define the Brouwer degree $d(\Psi, \mathcal{C}, U_{II}(1))$ for each target image $U_{II}(1)$, considering left data $U(0)$ as a fixed parameter [Bro11, Mil65, Hir94, Pra06, DM21]. Recall that Brouwer degree of a proper map is invariant under homotopy, and, for regular values $U_{II}(1)$, for which Ψ^{-1} consists of a finite set of isolated nondegenerate roots, is given by

$$(2.17) \quad d(\Psi, \mathcal{C}, U_{II}(1)) := \sum_{C_2 \in \Psi^{-1}(U_{II}(1))} \text{sgn det } d\Psi(C_2).$$

This includes the case $\Psi^{-1}(U_{II}(1)) = \emptyset$, for which $d(\Psi, \mathcal{C}, U_{II}(1)) = 0$, hence nonzero Brouwer degree implies existence of a solution.

By homotopy invariance, the degree may be computed at any such $U_{II}(1)$, in particular the value $U_{II}^*(1) = U_{II}(0)$ to which Theorem 2.11 applies. For this value, the inverse image of Ψ consists of a single C_2^* corresponding to a constant solution. Moreover, by Proposition 2.10, C_2^* is an isolated nondegenerate root, with $\text{sgn det } d\Psi(C_2^*) = +1$, hence by (2.17) $d(\Psi, \mathcal{C}, U_{II}^*(1)) = +1$. By homotopy invariance, therefore, $d(\Psi, \mathcal{C}, U_{II}(1)) = +1$ for any $U_{II}(1) \in \mathcal{U}^{int}$, implying existence of a solution by nonvanishing of the degree. $\square$

Remark 2.14. *The condition that Ψ be proper is quite strict in practice, as solutions may escape to infinity, etc. It was verified in [BMZ21] for the compressible Navier–Stokes equations with polytropic equation of state. It is a very interesting open question whether it holds for a general convex equation of state, or for more complicated physical systems such as viscoelasticity and magnetohydrodynamics (MHD).*

Remark 2.15. *It is worth noting that under solvability assumption (2.15), the shock tube problem (namely the Navier–Stokes equations in a tube) reduces to a nonlinear elliptic problem in U_{II} , with Dirichlet boundary conditions, to which all of the theory of global existence and uniqueness for such may be applied. However, up to now we have not been able to make use of this connection; rather our studies of specific systems seem to be an interesting source of examples for elliptic theory.*

2.4.3 Uniqueness and spectral stability

For large-amplitude solutions, neither uniqueness nor spectral stability appear to follow from the existence of a convex entropy. Indeed, numerical computations of [BMZ21] indicate that for the common example of the compressible Navier–Stokes equation with an artificially chosen convex equation of state—hence possessing a convex entropy—both failure of local uniqueness, and failure of spectral stability may occur, with steady and Hopf bifurcations.

2.5 Conclusions

For small-amplitude data, local existence, uniqueness, and spectral stability may be deduced from the standard structural assumptions of symmetrizability. For systems with convex entropy, we obtain, further, from global uniqueness of solutions of constant data. Provided the mapping Ψ may be shown to be proper (in the sense described above), we obtain as a corollary large-amplitude existence; however, this appears possibly quite restrictive. The (numerical) gas-dynamical examples of [BMZ21] suggest that large-amplitude uniqueness and spectral stability *in general do not hold* for systems possessing a convex entropy. Thus, these questions must apparently be addressed either by problem-specific analysis, asymptotic limit, or numerical Evans function evaluation as in [MZ19, BMZ21]

3 Asymptotic limits I : the inviscid limit in the scalar case

In this section we study the behavior of steady solutions of scalar viscous conservation laws. More precisely, we consider the problem

$$(3.1) \quad \begin{cases} \varepsilon \hat{u}_{xx}^\varepsilon = (f(\hat{u}^\varepsilon))_x, & 0 < x < 1, \\ \hat{u}^\varepsilon(0) = u_0, \hat{u}^\varepsilon(1) = u_1, \end{cases}$$

for $u_0, u_1 \in \mathbb{R}$ and $\varepsilon > 0$. We will assume in this section that

(A0) f is a $\mathcal{C}^2$ strictly convex function satisfying $f(0) = f'(0) = 0$.

We will also assume sometimes that

(A1) $f''(0) \neq 0$.

The second assumption is not necessary but we will suppose it sometimes by simplicity when we will deal with characteristic boundary layers and double boundary layers. Our goal is to give a description of these steady solutions especially when ε goes to 0. First, we give an existence result.

Proposition 3.1. *Assume that f is $\mathcal{C}^1(\mathbb{R})$. For any $\varepsilon > 0$ and any $u_0, u_1 \in \mathbb{R}$, there exists a unique solution of (3.1).*

In the following, we will denote by $\hat{u}^\varepsilon$ the unique solution of (3.1).

Proof. The arguments used are based on the proof of Proposition 2.1 in [MZ19]. We introduce the ODE with parameter $b \in \mathbb{R}$

$$\varepsilon y'_b = b + f(y_b), \quad y_b(0) = u_0$$

and denote by y_b the unique maximal solution. Note that y_b is monotonic. We define the set I_{u_0} as follows: $b \in I_{u_0}$ if and only if y_b is defined on $[0, 1]$. The set I_{u_0} is open and $-f(u_0) \in I_{u_0}$. Furthermore, by the comparison theorem for ODE, if $b_1 < b_2$, then $y_{b_1} < y_{b_2}$ on the intersection of the domains of definition of the two functions. Therefore I_{u_0} is an interval. Then, we introduce the map

$$\phi : b \in I_{u_0} \mapsto y_b(1) - u_1.$$

The comparison theorem for ODE asserts that ϕ is increasing. Furthermore, ϕ is not bounded from below and not bounded from above. Therefore, there exists a unique $b \in I_{u_0}$ such that $\phi(b) = u_1$. $\square$

The purpose of this section is to describe the behavior of $\hat{u}^\varepsilon$. If $u_0 = u_1$ then $\hat{u}^\varepsilon$ is constant. If $u_0 > u_1$, $\hat{u}^\varepsilon$ is decreasing and we talk about compressive waves. If $u_0 < u_1$, $\hat{u}^\varepsilon$ is increasing and we talk about expansive waves.

Before that, we would like to give a motivation of studying such solutions. We will show that there are spectrally stable (and then nonlinearly stable by adapting Section 6 of [MZ19]). We introduce the unbounded operator $\mathcal{L}$ on $L^2(0,1)$ with domain $\mathcal{D}(\mathcal{L}) = H^2(0,1) \cap H_0^1(0,1)$ as

$$\mathcal{L}v = \varepsilon v'' - (f'(\hat{u}^\varepsilon)v)'.$$

We denote by $\sigma(\mathcal{L})$ the spectrum of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$.

Lemma 3.2. *Assume that f is $\mathcal{C}^2(\mathbb{R})$. Let $\varepsilon > 0$ and $u_0, u_1 \in \mathbb{R}$. The inverse of $\mathcal{L}$ exists and is compact. The set $\sigma(\mathcal{L})$ only contains eigenvalues.*

Proof. The arguments used are based on the proof of Proposition 3.2 in [MZ19]. For any $g \in L^2(0,1)$ the system

$$\begin{cases} \varepsilon \hat{v}'' = (f'(\hat{u}^\varepsilon)v)' + g \\ v(0) = 0, v(1) = 0 \end{cases}$$

admits the unique solution

$$v(x) = \int_0^x \exp\left(-\int_y^x f'(\hat{u}^\varepsilon) \frac{dz}{\varepsilon}\right) \left[\int_0^y g(z) \frac{dz}{\varepsilon} - \frac{\exp\left(-\int_y^1 f'(\hat{u}^\varepsilon) \frac{dy}{\varepsilon}\right) \int_0^y g(z) \frac{dz}{\varepsilon}}{\exp\left(-\int_y^1 f'(\hat{u}^\varepsilon) \frac{dy}{\varepsilon}\right)} \right] dy$$

so that $\mathcal{L}$ is invertible. Furthermore, if g is in a ball of $L^2(0,1)$, the previous formula show that v is in a ball of $H_0^1(0,1)$ so that, by Rellich–Kondrachov theorem, $\mathcal{L}^{-1}$ is compact. $\square$

We can now state the spectral stability of $\hat{u}^\varepsilon$. The following proposition corresponds to Lemma 2.5 in [KK86].

Proposition 3.3. *Assume that f is $\mathcal{C}^2(\mathbb{R})$. Any eigenvalue of $\mathcal{L}$ is real and negative.*

3.1 Compressive waves

This subsection is devoted to the study of compressive waves, i.e. $u_0 > u_1$. The main result of this subsection asserts that $\hat{u}^\varepsilon$ is a piece of a whole-line steady shock wave. In the following, we define, for $y \neq 0$, the quantity $y^\dagger$ as the unique solution of the equation $f(x) = f(y)$ with $x \neq y$. We first recall some facts about whole-line shocks. Let $a > 0$. We introduce the steady shock ODE

$$(3.2) \quad \begin{cases} u'' = (f(u))', & x \in \mathbb{R}, \\ u(-\infty) = a, & u(+\infty) = a^\dagger. \end{cases}$$

This problem is invariant under translation so that it is convenient to impose the extra condition

$$(3.3) \quad u(0) = 0.$$

We recall some basic properties of the solution of this problem.

Lemma 3.4. *Assume that f satisfies (A0). For any $a > 0$, Problem (3.2)-(3.3) admits a unique solution u . The function u is decreasing. Furthermore, there exists a constant $C_1 > 0$ such that for any $x \leq 0$,*

$$|u(x) - a| + |u'(x)| \leq C_1 e^{f'(a)x},$$

and there exists a constant $C_2 > 0$ such that for any $x \geq 0$,

$$|u(x) - a^\dagger| + |u'(x)| \leq C_2 e^{f'(a^\dagger)x}.$$

In the following we denote by $u(\cdot; a)$ this unique solution. Note that $u(\cdot; \frac{a}{\varepsilon})$ is the unique solution of

$$\begin{cases} \varepsilon u'' = (f(u))', & x \in \mathbb{R}, \\ u(-\infty) = a, & u(+\infty) = a^\dagger. \end{cases}$$

Proof. We briefly recall the proof of this statement. It is convenient to consider the equivalent problem

$$(3.4) \quad \begin{cases} u' = f(u) - f(a), & x \in \mathbb{R}, \\ u(0) = 0. \end{cases}$$

Introducing the map

$$\phi_a : y \in (a^\dagger, a) \mapsto \int_0^y \frac{dz}{f(z) - f(a)} \in \mathbb{R},$$

we note that ϕ_a is decreasing and bijective. we define $u(x) := \phi_a^{-1}(x)$. Then, we notice that

$$(u')' = f'(u)u', \quad u'(0) = -f(a),$$

so that $u'(x) = -f(a) \exp\left(\int_0^x f'(u(y))dy\right)$. We get that u' and $u - a$ go exponentially to 0 as $x \rightarrow -\infty$. Therefore, the function $x \in \mathbb{R}^- \mapsto \int_0^x f'(u(y))dy - f'(a)x$ is bounded. The estimates on $\mathbb{R}^-$ follow easily. The estimates on $\mathbb{R}^+$ are similar. $\square$

The next lemma provides an estimate of the differential of $u(\cdot; a)$ with respect to a .

Lemma 3.5. *Assume that f satisfies (A0). Let $a > 0$ and $b \in \mathbb{R}$. Denote $u = u(\cdot; a)$ and $v_b = \lim_{s \rightarrow 0} \frac{u(\cdot; a + sb) - u(\cdot; a)}{s}$. Then, there exists a constant $C > 0$ depending only on a such that, for any $x \in \mathbb{R}$,*

$$|v_b(x)| \leq C|b|.$$

Proof. The function v satisfies the ODE

$$v'_b = f'(u)v_b - f'(a)b, \quad v(0) = 0,$$

so that $v_b(x) = -bf'(a) \int_0^x \exp\left(\int_y^x f'(u)dz\right) dy$. The estimate follows from the estimates of Lemma 3.4. $\square$

We get from Lemma 3.4 that for any $a > 0$, for any $y \in (a^\dagger, a)$, the equation $u(x; a) = y$ has a unique solution. We keep the previous notation and denote as $\phi_a(y)$ this unique solution. We can now state the main result of this section.

Theorem 3.6. *Assume that f satisfies (A0). Let $\varepsilon > 0$, $u_0, u_1 \in \mathbb{R}$ such that $u_0 > u_1$. Then the unique solution $\hat{u}^\varepsilon$ of Problem (3.1) is the restriction to $[0, 1]$ of a whole-line shock: there exists a unique pair $(a, \lambda) \in \mathbb{R}_*^+ \times \mathbb{R}$ such that, for any $x \in [0, 1]$, we have $\hat{u}^\varepsilon(x) = u\left(\frac{x}{\varepsilon} + \lambda; a\right)$. Furthermore,*

- 1) *If $f(u_0) > f(u_1)$, the shock moves to the endpoint $x = 1$ as $\varepsilon \rightarrow 0$: $u_0 > 0$ and there exists a constant $C > 0$ such that, for any $x \in [0, 1]$,*

$$\left| \hat{u}^\varepsilon(x) - u\left(\frac{x-1}{\varepsilon} + \phi_{u_0}(u_1); u_0\right) \right| \leq C \exp\left(-\frac{f'(u_0)}{\varepsilon}\right).$$

- 2) *If $f(u_0) < f(u_1)$, the shock moves to the endpoint $x = 0$ as $\varepsilon \rightarrow 0$: $u_1 < 0$ and there exists a constant $C > 0$ such that, for any $x \in [0, 1]$,*

$$\left| \hat{u}^\varepsilon(x) - u\left(\frac{x}{\varepsilon} + \phi_{u_1^\dagger}(u_0); u_1^\dagger\right) \right| \leq C \exp\left(\frac{f'(u_1)}{\varepsilon}\right).$$

- 3) *If $f(u_0) = f(u_1)$, the shock stays inside the interval as $\varepsilon \rightarrow 0$: $u_0 > 0 > u_1 = u_0^\dagger$ and there exists a constant $C > 0$ such that, for any $x \in [0, 1]$,*

$$\left| \hat{u}^\varepsilon(x) - u\left(\frac{x - \frac{f'(u_0^\dagger)}{f'(u_0^\dagger) - f'(u_0)}}{\varepsilon} + b; u_0\right) \right| \leq \frac{C}{\varepsilon} \exp\left(-\frac{f'(u_0^\dagger)f'(u_0)}{f'(u_0^\dagger) - f'(u_0)} \frac{1}{\varepsilon}\right).$$

with

$$b = \frac{1}{f'(u_0^\dagger) - f'(u_0)} \left(\int_{u_0^\dagger}^0 \frac{f'(u_0^\dagger) - f'(z)}{f(u_0) - f(z)} dz + \int_0^{u_0} \frac{f'(u_0) - f'(z)}{f(u_0) - f(z)} dz \right).$$

Remark 3.7. *Note that the previous theorem also works with characteristic endpoints, i.e. the cases $u_0 = 0 > u_1$ or $u_0 > u_1 = 0$. Therefore, in these cases, one expects a boundary layer of size ε whereas characteristic boundary layers on the half-line have size $\sqrt{\varepsilon}$.*

Remark 3.8. *When f is even, we have $u_0^\dagger = -u_0$ and in the case $u_1 = -u_0$ a better look of the proof shows that*

$$\left| \hat{u}^\varepsilon(x) - u\left(\frac{x - \frac{1}{2}}{\varepsilon}; u_0\right) \right| \leq C \exp\left(-\frac{f'(u_0)}{2\varepsilon}\right).$$

Proof. We consider first the case $f(u_0) > f(u_1)$. In that case $u_0 > 0$ and $u_1 \in (u_0^\dagger, u_0)$. We solve the following system, with unknowns $(a, \lambda) \in (u_0, +\infty) \times \mathbb{R}^-$,

$$\begin{cases} u(\lambda; a) = u_0 \\ u\left(\frac{1}{\varepsilon} + \lambda; a\right) = u_1 \end{cases} \iff \begin{cases} \lambda = \phi_a(u_0) \\ \frac{1}{\varepsilon} + \phi_a(u_0) = \phi_a(u_1). \end{cases}$$

Then, we remark that $\psi := a \in (u_0, +\infty) \mapsto \phi_a(u_1) - \phi_a(u_0) \in \mathbb{R}_*^+$ is actually

$$\psi(a) = \int_{u_1}^{u_0} \frac{dz}{f(a) - f(z)}$$

and this map is continuous and decreasing, $\lim_{a \rightarrow u_0^+} \psi(a) = +\infty$ and $\lim_{a \rightarrow +\infty} \psi(a) = 0$. That proves that $\hat{u}^\varepsilon$ is the restriction to $[0, 1]$ of a unique whole-line shock. Furthermore, we remark that $\lim_{\varepsilon \rightarrow 0} a = u_0$. Then, denoting by simplicity f^{-1} as the inverse function of $f|_{\mathbb{R}^+}$, we have

$$\begin{aligned} \int_0^{u_0} \frac{dz}{f(a) - f(z)} - \frac{1}{f'(u_0)} \int_0^{f(u_0)} \frac{dy}{f(a) - y} &= \int_0^{f(u_0)} \frac{1}{f'(f^{-1}(y))f'(u_0)} \frac{f'(u_0) - f'(f^{-1}(y))}{f(a) - y} dy \\ &= \frac{1}{f'(u_0)} \int_0^{u_0} \frac{f'(u_0) - f'(z)}{f(a) - f(z)} dz \end{aligned}$$

the last expression being bounded uniformly with respect to $\varepsilon \rightarrow 0$. Therefore, there exists a constant $C > 0$ such that

$$0 \leq f'(u_0)(a - u_0) \leq f(a) - f(u_0) \leq C e^{-\frac{f'(u_0)}{\varepsilon}}.$$

That proves that $a = u_0 + \mathcal{O}(e^{-\frac{f'(u_0)}{\varepsilon}})$. Furthermore, we have

$$\lambda = -\frac{1}{\varepsilon} + \phi_a(u_1) = -\frac{1}{\varepsilon} + \phi_{u_0}(u_1) + \mathcal{O}(e^{-\frac{f'(u_0)}{\varepsilon}}).$$

Finally, for any $x \in [0, 1]$, using Lemma 3.5 and the fact $u'(\cdot; u_0)$ is bounded, there exists a constant $C > 0$ such that

$$\begin{aligned} \left| u\left(\frac{x}{\varepsilon} + \lambda; a\right) - u\left(\frac{x-1}{\varepsilon} + \phi_{u_0}(u_1); a\right) \right| &\leq \left| u\left(\frac{x}{\varepsilon} + \lambda; a\right) - u\left(\frac{x}{\varepsilon} + \lambda; u_0\right) \right| \\ &\quad + \left| u\left(\frac{x}{\varepsilon} + \lambda; u_0\right) - u\left(\frac{x-1}{\varepsilon} + \phi_{u_0}(u_1); u_0\right) \right| \\ &\leq C |a - u_0| + C \left| \lambda + \frac{1}{\varepsilon} + \phi_{u_0}(u_1) \right|. \end{aligned}$$

The case $f(u_0) < f(u_1)$ follows similarly. If now $f(u_0) = f(u_1)$, following the previous argument, there exists a constant $C > 0$ independent of $\varepsilon \rightarrow 0$ such that

$$\left| \int_{u_0^\dagger}^{u_0} \frac{dz}{f(a) - f(z)} - \frac{1}{f'(u_0)} \int_0^{f(u_0)} \frac{dy}{f(a) - y} - \frac{1}{f'(u_0^\dagger)} \int_{f(u_0)}^0 \frac{dy}{f(a) - y} \right| \leq C$$

so that $a = u_0 + \mathcal{O}\left(e^{-\frac{f'(u_0^\dagger)f'(u_0)}{f'(u_0^\dagger)-f'(u_0)}\frac{1}{\varepsilon}}\right)$. Then, we get after some computations that

$$\begin{aligned} \lambda = & -\frac{f'(u_0^\dagger)}{f'(u_0^\dagger)-f'(u_0)}\frac{1}{\varepsilon} + \frac{1}{f'(u_0^\dagger)-f'(u_0)}\left(\int_{u_0^\dagger}^0 \frac{f'(u_0^\dagger)-f'(z)}{f(u_0)-f(z)}dz + \int_0^{u_0} \frac{f'(u_0)-f'(z)}{f(u_0)-f(z)}dz\right) \\ & + \frac{1}{f'(u_0^\dagger)-f'(u_0)}c_a \end{aligned}$$

with

$$c_a = \int_{u_0^\dagger}^0 \frac{f'(u_0^\dagger)-f'(z)}{f(u_0^\dagger)-f(z)} \frac{f(u_0)-f(a)}{f(a)-f(z)}dz + \int_0^{u_0} \frac{f'(u_0)-f'(z)}{f(u_0)-f(z)} \frac{f(u_0)-f(a)}{f(a)-f(z)}dz.$$

It is easy to check that there exists a constant $C > 0$ such that

$$|c_a| \leq C(a - u_0)(1 + |\ln(a - u_0)|)$$

and the result follows from the same way as the previous cases. $\square$

3.2 Expansive waves

This subsection is devoted to the study of expansive waves, i.e. $u_0 < u_1$. We will consider two different cases. Firstly, we will study the case where u_0 and u_1 agree in sign. In that case, we obtain, as $\varepsilon \rightarrow 0$, a single boundary layer. Secondly, we will consider the situation where $u_0 < 0 < u_1$. We will see in that case that a double boundary layer occurs as $\varepsilon \rightarrow 0$.

3.2.1 Single boundary layer at $x = 1$

We start with the case $0 \leq u_0 < u_1$. The flow is outgoing at $x = 1$ and we expect a boundary layer at $x = 1$. Influence by the work of Gisclon-Serre [GS94, Gis96] on the half-line, we consider the following expansion, for any $x \in [0, 1]$,

$$(3.5) \quad \hat{u}^\varepsilon(x) = u_0 + v\left(\frac{1-x}{\varepsilon}\right) + w\left(\frac{1-x}{\varepsilon}\right)$$

where v is a function defined on $\mathbb{R}^+$ that satisfies the following ODE

$$(3.6) \quad v'' = -(f(u_0 + v))', \quad v(0) = u_1 - u_0 \text{ and } v(+\infty) = 0,$$

and w a function defined on $[0, \frac{1}{\varepsilon}]$ supposed to be small and satisfying $w(0) = 0$. We begin our analysis with a technical lemma about the solution of Problem (3.6).

Lemma 3.9. *Assume that f satisfies (A0). Let $u_0, u_1 \in \mathbb{R}$ such that $0 \leq u_0 < u_1$. Assume that v solves (3.6). Then v is positive, v' is negative. There exists a constant $C > 0$ such that, for any $y \in \mathbb{R}^+$, if $u_0 > 0$ we have*

$$|v(y)| \leq Ce^{-f'(u_0)y},$$

whereas if $u_0 = 0$ and f satisfies (A1) we have

$$|v(y)| \leq \frac{C}{1+y}.$$

Proof. The proof is similar to the one of Lemma 3.4. We consider the equivalent problem by integrating from $+\infty$ to $y \geq 0$

$$v' = f(u_0) - f(u_0 + v), \quad v(0) = u_1 - u_0.$$

Introducing the map

$$\phi : y \in (0, u_1 - u_0] \mapsto \int_y^{u_1 - u_0} \frac{dz}{f(u_0 + y) - f(u_0)} \in \mathbb{R}_+,$$

we note that ϕ is decreasing and bijective. Then, $v(x) = \phi^{-1}(x)$. If $u_0 > 0$, the estimate follows with the same argument as in the proof of Lemma 3.4 whereas if $u_0 = 0$ we use the fact that under assumption (A1) there exists a constant $\alpha > 0$ such that $f(y) \geq \alpha y^2$ for $y > 0$ small enough. $\square$

We can now state the main result of this subsubsection.

Theorem 3.10. *Assume that f satisfies (A0). Let $\varepsilon \in (0, 1]$, $u_0, u_1 \in \mathbb{R}$ such that $0 \leq u_0 < u_1$. Let $\hat{u}^\varepsilon$ be the unique solution of Problem (3.1) and v the unique solution of Problem (3.6). Then, there exists a constant $C > 0$, such that for any $x \in [0, 1]$, if $u_0 > 0$ we have*

$$\left| \hat{u}^\varepsilon(x) - u_0 - v\left(\frac{1-x}{\varepsilon}\right) \right| \leq Ce^{-\frac{f'(u_0)}{\varepsilon}},$$

whereas if $u_0 = 0$ and f satisfies (A1) we have

$$\left| \hat{u}^\varepsilon(x) - v\left(\frac{1-x}{\varepsilon}\right) \right| \leq C\varepsilon.$$

Note that in the characteristic case, we lose the exponential decay.

Proof. We use the expansion (3.5) and notice that w satisfies

$$\begin{cases} w'' = (f'(u_0 + v) - f'(u_0 + v + w))v' - f'(u_0 + v + w)w', & \text{on } [0, \frac{1}{\varepsilon}], \\ w(0) = 0, \quad w(\frac{1}{\varepsilon}) = -v(\frac{1}{\varepsilon}). \end{cases}$$

Since f' is increasing and $v' < 0$, the maximum principle gives that $-v(\frac{1}{\varepsilon}) \leq w \leq 0$. The result follows. $\square$

3.2.2 Single boundary layer at $x = 0$

We study the case $u_0 < u_1 \leq 0$. The flow is outgoing at $x = 0$ and we expect a boundary layer at $x = 0$. Here we consider the following expansion, for any $x \in [0, 1]$,

$$\hat{u}^\varepsilon(x) = u_1 + v\left(\frac{x}{\varepsilon}\right) + w\left(\frac{x}{\varepsilon}\right)$$

where v is a function defined on $\mathbb{R}^+$ that satisfies the following ODE

$$(3.7) \quad v'' = (f(u_1 + v))', \quad v(0) = u_0 - u_1 \text{ and } v(+\infty) = 0,$$

and w a function defined on $[0, \frac{1}{\varepsilon}]$ supposed to be small and satisfying $w(0) = 0$. We have similar results as in the previous subsection.

Lemma 3.11. *Assume that f satisfies (A0). Let $u_0, u_1 \in \mathbb{R}$ such that $u_0 < u_1 \leq 0$. Assume that v solves (3.7). Then v is negative, v' is positive. There exists a constant $C > 0$ such that, for any $y \in \mathbb{R}^+$, if $u_1 < 0$ we have*

$$|v(y)| \leq Ce^{f'(u_1)y},$$

whereas if $u_1 = 0$ and f satisfies (A1) we have

$$|v(y)| \leq \frac{C}{1+y}.$$

Theorem 3.12. *Assume that f satisfies (A0). Let $\varepsilon \in (0, 1]$, $u_0, u_1 \in \mathbb{R}$ such that $u_0 < u_1 \leq 0$. Let $\hat{u}^\varepsilon$ be the unique solution of Problem (3.1) and v the unique solution of Problem (3.7). Then, there exists a constant $C > 0$, such that for any $x \in [0, 1]$, if $u_1 < 0$ we have*

$$\left| \hat{u}^\varepsilon(x) - u_1 - v\left(\frac{x}{\varepsilon}\right) \right| \leq Ce^{\frac{f'(u_1)}{\varepsilon}},$$

whereas if $u_1 = 0$ and f satisfies (A1) we have

$$\left| \hat{u}^\varepsilon(x) - v\left(\frac{x}{\varepsilon}\right) \right| \leq C\varepsilon.$$

3.2.3 Double boundary layer

We finally study the case where $u_0 < 0 < u_1$. Here the two endpoints are outgoing. We expect then a boundary layer on both side: a double boundary layer. We first need a technical lemma on the unique solution of (3.1).

Lemma 3.13. *Assume that f satisfies (A0) and (A1). Let $u_0, u_1 \in \mathbb{R}$ such that $u_0 < 0 < u_1$. Assume that $\hat{u}^\varepsilon$ is the unique solution of (3.1). Then, there exists a constant $C > 0$ such that*

$$\left| \hat{u}^\varepsilon\left(\frac{1}{2}\right) \right| \leq C\varepsilon.$$

Proof. From Assumptions (A0)-(A1), there exists two constants $\alpha, \beta > 0$ so that for any $y \in (u_0, u_1)$,

$$(3.8) \quad \alpha|y|^2 \leq f(y) \leq \beta|y|^2.$$

Then, we introduce x_ε the unique element of $(0, 1)$ such that $\hat{u}^\varepsilon(x_\varepsilon) = 0$. Integrating (3.1), we get

$$\varepsilon \hat{u}_x^\varepsilon = \varepsilon \hat{u}_x^\varepsilon(x_\varepsilon) + f(\hat{u}^\varepsilon).$$

Note also that $\hat{u}_x^\varepsilon(x_\varepsilon) = \min \hat{u}_x^\varepsilon$ so that $u_1 - u_0 \geq \hat{u}_x^\varepsilon(x_\varepsilon) > 0$. Using (3.8), we easily obtain from the previous ODE that

$$\left(\arctan \left(\sqrt{\frac{\beta}{\varepsilon \hat{u}_x^\varepsilon(x_\varepsilon)}} \hat{u}^\varepsilon \right) \right)' \leq \sqrt{\frac{\beta \hat{u}_x^\varepsilon(x_\varepsilon)}{\varepsilon}}$$

and integrating from 0 to 1 and using that $\varepsilon \hat{u}_x^\varepsilon(x_\varepsilon)$ tends to 0, we prove the existence of $m > 0$ such that $\hat{u}_x^\varepsilon(x_\varepsilon) \geq m\varepsilon$. Then, we similarly obtain

$$\left(\arctan \left(\sqrt{\frac{\alpha}{\varepsilon^2 m}} \hat{u}^\varepsilon \right) \right)' \geq \sqrt{\alpha m}$$

and integrating from 0 to $\frac{1}{2}$ and from $\frac{1}{2}$ to 1 we get the desired estimate. $\square$

Then, we introduce the two boundary layers: let v_0 function defined on $\mathbb{R}^+$ that satisfies the ODE

$$(3.9) \quad v_0'' = (f(v_0))', \quad v_0(0) = u_0 \text{ and } v_0(+\infty) = 0,$$

and let v_1 function defined on $\mathbb{R}^+$ that satisfies the ODE

$$(3.10) \quad v_1'' = -(f(v_1))', \quad v_1(0) = u_1 \text{ and } v_1(+\infty) = 0.$$

We can now state the main result of this subsection.

Theorem 3.14. *Assume that f satisfies (A0) and (A1). Let $\varepsilon \in (0, 1]$, $u_0, u_1 \in \mathbb{R}$ such that $u_0 < 0 < u_1$. Let $\hat{u}^\varepsilon$ be the unique solution of Problem (3.1) and v_0 and v_1 the unique solution of respectively Problem (3.9) and Problem (3.10). Then, there exists a constant $C > 0$, for any $x \in [0, 1]$,*

$$\left| \hat{u}^\varepsilon(x) - v_0\left(\frac{x}{\varepsilon}\right) - v_1\left(\frac{1-x}{\varepsilon}\right) \right| \leq C\varepsilon.$$

Proof. Recall that Lemmas 3.9 and 3.11 shows that for any $y > 0$

$$0 < -v_0(y) \lesssim \frac{1}{1+y}, \quad 0 < v_1(y) \lesssim \frac{1}{1+y}, \quad v_0' > 0, \quad v_1' < 0.$$

We first consider what happens on $[0, \frac{1}{2}]$. Decomposing $\hat{u}^\varepsilon$ on $[0, \frac{1}{2}]$ as

$$\hat{u}^\varepsilon(x) = v_0\left(\frac{x}{\varepsilon}\right) + w\left(\frac{x}{\varepsilon}\right)$$

we get, as in Subsection 3.2.2 that

$$\begin{cases} w'' = (f'(v_0 + w) - f'(v_0))v_0' + f'(v_0 + w)w', & \text{on } [0, \frac{1}{2\varepsilon}], \\ w(0) = 0, \quad w(\frac{1}{2\varepsilon}) = \hat{u}^\varepsilon(\frac{1}{2}) - v_0(\frac{1}{2\varepsilon}). \end{cases}$$

Similarly to the proof of Theorem 3.10, we get $0 \leq |w| \leq |\hat{u}^\varepsilon(\frac{1}{2}) - v_0(\frac{1}{2\varepsilon})|$ from the maximum principle. Using the previous lemma, the desired control on $[0, \frac{1}{2}]$ follows since $|v_0(\frac{1}{2\varepsilon})| \lesssim \varepsilon$ and for any $x \in [0, \frac{1}{2}]$, $|v_1(\frac{1-x}{\varepsilon})| \leq |v_1(\frac{1}{2\varepsilon})| \lesssim \varepsilon$. A similar proof can be done on $[\frac{1}{2}, 1]$. $\square$

4 Asymptotic limits II : General case

4.1 Small-viscosity/large interval asymptotics

In either the vanishing-viscosity limit, or the large-interval limit $[0, X]$, $X \rightarrow +\infty$ after rescaling back to the unit interval $[0, 1]$, we are led to consider in place of (1.1)

$$(4.1) \quad \partial_t f^0(U) + f(U)_x = \varepsilon (B(U)U_x)_x, \quad 0 < x < 1,$$

with $\varepsilon = \frac{1}{X}$, $\varepsilon \rightarrow 0^+$, and the steady profile equation is

$$(4.2) \quad f(U)' = \varepsilon (B(U)U')'.$$

Formally setting $\varepsilon = 0$ in (4.2), we obtain $f(U) \equiv \text{constant}$, or $U \equiv \text{constant}$ on smooth portions, separated by standing shock and boundary layers. This indicates a rich “zoo” of possible steady solution structures. Some examples from the isentropic gas dynamics case are displayed in Figure 1 which corresponds to behaviors we observed in Section 3.

4.1.1 Hyperbolic structure

It is readily deduced that the limiting configurations depicted in Figure 1 are in fact the only possibilities for the isentropic case. For, $\rho u \equiv \text{constant}$ imposes $\rho, u > 0$ throughout the limiting pattern, whence all states have either one or two positive characteristics $\alpha = u \pm c$, where c is sound speed. This in turn implies, by general results of [MZ03] that, as rest points of the scalar steady profile ODE, they are attractors or repellers, respectively. A nontrivial nondegenerate boundary layer at the left endpoint $x = 0$ must terminate at $x = 0^+$ at a rest point, which must therefore be an attractor; at the right endpoint, $x = 1^-$ a repeller. Interior shocks must connect a repeller on the left with a saddle on the right. It follows that nontrivial boundary layers and interior shocks cannot coexist, but occur only separately. Moreover, there can occur at most one nondegenerate boundary layer, either at the left or the right endpoint. The final possibility completing our zoo of possible configurations

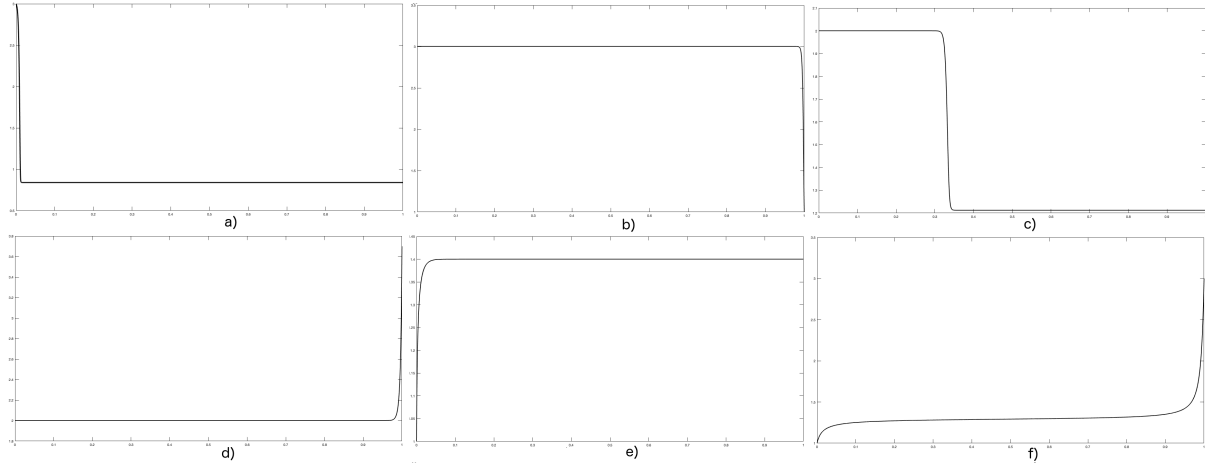


Figure 1: Steady solutions of the Navier-Stokes isentropic equations (4.3) on $(0,1)$ with viscosity $\nu = 0.01$ and $p(\rho) = \rho^{1.4}$. We only plot u . Panels a) and b) depict left and right compressive boundary layers, c) an interior shock, and d) and e) left and right expansive boundary layers, each connecting to nondegenerate rest states of the steady profile ODE, hence exponentially convergent. Panel f) depicts a double boundary layer (expansive) consisting of left and right degenerate (hence non-exponentially convergent) boundary layers meeting at a characteristic middle state.

is a double boundary layer configuration, for which the end point must necessarily be degenerate, corresponding to a “sonic”, or “characteristic” point where $u = c$. Shocks or boundary layers connecting to a nondegenerate rest point decay exponentially; those connecting to a degenerate rest point decay algebraically. For further discussion of boundary layer structure for the compressible Navier-Stokes equations, see, e.g., [SZ01, GMWZ05].

For the nonisentropic case, we again have $\rho, u > 0$, imposing in this instance that state have either two or three positive characteristics $\alpha = u - c, u, u + c$; as rest points of the steady profile equation, these correspond to saddle points or repellers, respectively. Similar analysis to the above yields again that left boundary layers and shocks cannot coincide; however, there is the new possibility of patterns consisting of a left boundary layer plus a right boundary layer, or an interior shock plus a right boundary layer, as nontrivial right boundary layers may connect to either repellers or saddles in the nonisentropic case.

4.1.2 Rigorous asymptotics

A very interesting open problem is to carry out the zero-viscosity limit rigorously, in preparation for the more complicated dynamics of the 2d shock tube problem. One might hope also to understand the spectra of such wave patterns as the approximate direct sum of the spectra of component layers, as would follow, for example, by the methods of [Zum10, Zum11] if the components layers remained appropriately spatially separated in the limit. A first

apparently nontrivial step, of interest in its own right, is to show for given boundary data existence and uniqueness of feasible limiting patterns as described in Section 4.1.1.

We illustrate this for the (surprisingly complicated) case of isentropic gas dynamics:

$$(4.3) \quad \begin{aligned} \rho_t + (\rho u)_x &= 0, & (\rho u)_t + (\rho u^2 + p(\rho))_x &= \nu u_{xx}; & \rho &> 0, & p', p'' > 0. \\ \rho(0) &= \rho_0, & u(0) &= u_0, & u(1) &= u_1. \end{aligned}$$

Proposition 4.1. *For isentropic gas dynamics (4.3), there exists, for each boundary data (ρ_0, u_0, u_1) , a unique limiting steady configuration consisting of: (i) a single boundary layer at left or right endpoint; (ii) a single interior standing shock; or (iii) a double boundary-layer connecting from each boundary to a “characteristic point” $u = c$, with $c = \sqrt{p'(\rho)}$ denoting sound speed. These determine a unique inviscid solution up to location of the interior shock. The unique viscous steady solution converges to an inviscid one with particular interior shock location as viscosity $\nu \rightarrow 0$ at sharp rates $\nu^{1/p}$ in L^p , $1 < p < \infty$ and $\nu \log \nu$ in L^1 .*

Proof. We fix (ρ_0, u_0) . Referring to Section 2 in [MZ19], a steady solution $(\hat{\rho}, \hat{u})$ of (4.3) is given by

$$(4.4) \quad m\nu\hat{\rho}' = \hat{\rho}^2(b - \psi(\hat{\rho})), \quad \hat{u} = m/\hat{\rho},$$

where ν is coefficient of viscosity; $m = \rho_0 u_0 > 0$; $\psi(\rho) := m^2/\rho + p(\rho)$ is convex, with limit $+\infty$ as $\rho \rightarrow 0$ and $\rho \rightarrow +\infty$; and b is a constant determined implicitly by $\hat{\rho}(1) = m/u_1$.

Evidently, ψ is minimized at the characteristic point ρ_* where $m^2/\rho_*^2 = p'(\rho_*)$. Note that by denoting $u_* = m/\rho_*$ we get $u_* = c$ by positivity of u_* and c . For $b > \psi_* := \psi(\rho_*)$, there exist nondegenerate rest points $\rho_- < \rho_* < \rho_+$, with $\hat{\rho}' > 0$ for $\rho_- < \hat{\rho} < \rho_+$ and $\hat{\rho}' < 0$ for $\hat{\rho} < \rho_-$ and $\hat{\rho} > \rho_+$. For $b = \psi_*$, there is a single degenerate rest point at $\hat{\rho} = \rho_*$, with $\hat{\rho}' < 0$ for all $\hat{\rho} \neq \rho_*$. Checking $\psi''(\rho_*) = 2m^2/\rho_*^3 + p''(\rho_*) > 0$, hence to lowest order the flow near ρ_* is of Riccati type, $\hat{\rho}' \sim -\nu^{-1}\hat{\rho}^2$. For $b < \psi_*$, there are no rest points and $\hat{\rho}'$ is uniformly negative. See Figure 2 for a typical phase portrait.

(case $\rho(0) \geq \rho(1)$) It follows that for any $\rho(0) > \rho(1) > \rho_*$, there exists a unique decreasing right viscous boundary layer with value $\rho(1)$ at $x = 1$ and converging at exponential rate $e^{-r_1|x|/\nu}$ as $x \rightarrow -\infty$ to the lefthand state $\rho(0)$. Similarly, for any $\rho_* > \rho(0) > \rho(1)$, there is a unique decreasing left viscous boundary layer with value $\rho(0)$ at $x = 0$ and converging as $e^{-r_0|x|/\nu}$ as $x \rightarrow +\infty$ to $\rho(1)$. For $\rho(0) \geq \rho_* \geq \rho(1)$, there are unique decreasing *characteristic* right and left viscous boundary layers connecting $\rho(1)$ and $\rho(0)$ to the characteristic point ρ_* , with convergence at *algebraic rate* $|x/\nu|^{-1}$.

(case $\rho(0) \leq \rho(1)$) Similarly, for $\rho(0) < \rho(1) < \rho_*$, there is a unique increasing and exponentially converging right viscous boundary layer from $\rho(1) < \rho_+(b)$ to $\rho(0) = \rho_-(b)$ for some $b > \psi_*$, while for $\rho_* < \rho(0) < \rho(1)$, there is a unique increasing and exponentially converging left viscous boundary layer from $\rho(0) > \rho_-(b)$ to $\rho(1) = \rho_+(b)$ for some $b > \psi_*$. For $\rho(0) < \rho_* < \rho(1)$, finally, there exists either a unique increasing and exponentially converging right viscous boundary layer from $\rho(1) < \rho_+(b)$ to $\rho(0) = \rho_-(b)$ for some $b > \psi_*$, a unique increasing and exponentially converging left viscous boundary layer from $\rho(0) >$

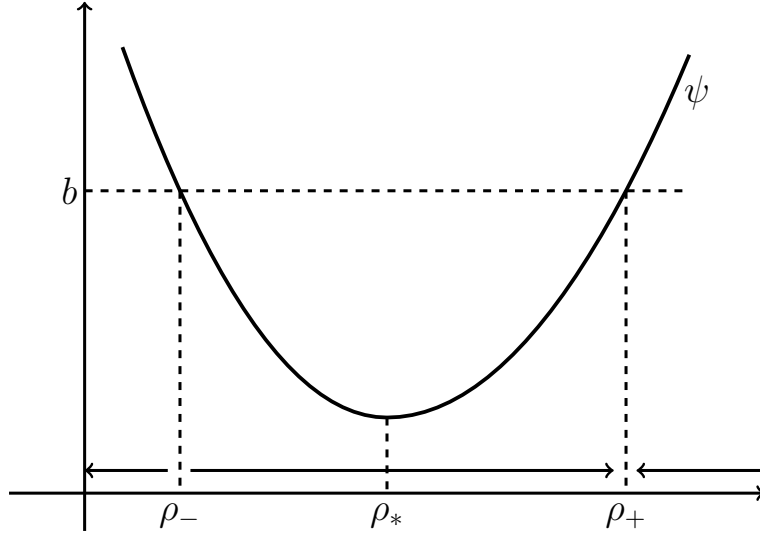


Figure 2: Phase portrait of the ODE (4.4)

$\rho_-(b)$ to $\rho(1) = \rho_+(b)$ for some $b > \psi_*$, or an exponentially converging stationary interior shock from $\rho(0) = \rho_-(b)$ to $\rho(1) = \rho_+(b)$ for some $b > \psi_*$.

Note, at the hyperbolic (inviscid, $\nu = 0$) level, that left and right characteristic boundary layers may be concatenated to form a “double boundary layer” with intermediate characteristic state ρ_* , connecting $\rho(0) > \rho_*$ to $\rho(1) < \rho_*$. In all other cases, the data $\rho(0)$ and $\rho(1)$ may be connected by a single element consisting of a left or right, increasing or decreasing boundary layer, or, in the limiting case, a single stationary interior shock.

Combining these elements, we obtain a unique feasible inviscid limit, up to arbitrary positioning of the interior standing shock. A refined analysis taking into account exponential decay rates $e^{-r_0|x|/\nu}$ and $e^{-r_1|x|/\nu}$ of the viscous shock as $x \rightarrow -\infty$ and $x \rightarrow +\infty$, respectively, combined with the linearization $[(\bar{u}^2 - \bar{c}^2)\Delta\rho] = 0$ of Rankine-Hugoniot relation $[m^2/\rho + p(\rho)] = 0$, where $[\cdot]$ denotes jump across the shock and $\Delta\rho$ denotes change in ρ due to truncation of the (infinite-extent) viscous shock at boundaries $x = 0, 1$, we obtain asymptotics $r_0x_s = r_1(1 - x_s) + O(\nu)$, or

$$(4.5) \quad x_s(\nu) = \frac{r_1}{r_0 + r_1} + O(\nu)$$

as $\nu \rightarrow 0$ toward the limiting shock location $x_s(0) = r_0/(r_0 + r_1)$. Similarly, the single left- or right-boundary layer solutions must be adjusted slightly due to exponential tails, in order to reach the correct value at the finite endpoint of the interval. We omit the details.

Putting these pieces back together, we obtain convergence of viscous to inviscid solution in L^1 at rate $O(\nu)$ for solutions containing shocks and noncharacteristic boundary layers. The analysis of double layer solutions formed by concatenation of characteristic left and right boundary layers is somewhat more involved. Noting that $\psi \sim (\rho - \rho_*)^2$ near ρ_*

in the boundary layer profile ODE, and splitting into regions where $(\rho - \rho_*)^2 \lesssim b$ and $(\rho - \rho_*)^2 \gtrsim b$, we find after a brief calculation that $|\rho - \rho_*|$ varies from 0 to $\sqrt{b} \sim \nu$ on an x -interval of order-one length and afterward grows like ν/x , giving a total L^1 convergence error $\sim \nu + \int_\nu^1 (\nu/x) dx \sim \nu \log \nu$. Other L^p norms, $1 < p < \infty$, go similarly. $\square$

4.1.3 Hyperbolic dynamics

A very interesting further question is the hyperbolic description of *dynamics* in the inviscid limit, in particular whether one may extend the formal description of Gisclon–Serre [GS94, Gis96, GMMZ09] for the half-line $[0, +\infty)$, based on viscous profiles, to the case of a bounded interval. In this approach, one imposes on the interior of the set the usual hyperbolic description, with jump conditions given by the requirement that there exist a corresponding viscous shock profile connecting the endstates of the jump. Analogously, one requires at the boundary that solutions lie in the “reachable set” of limiting states as $x \rightarrow +\infty$ of a viscous boundary layer profile satisfying the viscous boundary conditions at $x = 0$. For *noncharacteristic* boundary layers, this description may be rigorously validated under very general circumstances [Gis96, GMMZ09], both in the sense that it provides a well-posed hyperbolic problem, and that the solution of this problem is the limit of viscous solutions.

In the present, bounded interval case, the appearance of characteristic boundary layers complicates matters. For illustration, considering again the case of isentropic gas dynamics (4.3) with viscous inflow/outflow boundary conditions specifying

$$(4.6) \quad (\hat{\rho}(0), \hat{u}(0), \hat{u}(1)) = (\rho_0, u_0, u_1); \quad \rho_0, u_0, u_1 > 0.$$

Then, the classification of boundary layers in the proof of Proposition 4.1 shows that the left inviscid boundary conditions induced by “reachability” criterion of Gisclon–Serre are given for “supersonic” states $\rho(0) \leq \rho_*$ by the full Dirichlet conditions

$$(4.7) \quad (\rho(0), u(0)) = (\rho_0, u_0), \quad \rho_0 \leq \rho_*,$$

and for “subsonic” states $u(0) \leq \sqrt{p'(\rho(0))}$ by

$$(4.8) \quad \rho(0)u(0) = m_0 := \rho_0 u_0, \quad \rho(0) \geq \rho_0^\dagger,$$

where $\rho^\dagger$ denotes the conjugate point connected to ρ by a standing viscous shock profile with $\rho u = m_0$. Here, $\rho(0), u(0), u(1)$ refer to interior hyperbolic solutions.

Different from the small-amplitude case treated in [GS94], both of these conditions are on a restricted range depending on ρ_0 , outside of which states are *disallowed*. Note, if ρ_0 and $\rho(0)$ are nearby and far from ρ_* , then $\rho(0) < \rho_*$ implies $\rho_0 < \rho_*$, necessarily. Similarly, $\rho(0)$ subsonic implies $\rho_0 > \rho_*$, hence $\rho_0^\dagger \leq \rho_*$ and so $\rho(0) \geq \rho_* \geq \rho_0^\dagger$ always. Thus, the restrictions in (4.7)(ii) and (4.8)(ii) are only for large-amplitude perturbations, so long as all states are bounded away from characteristic values.

Meanwhile, for subsonic states $u(1) \leq \sqrt{p'(\rho(1))}$ (which serve only as repellors in backward x flow, so admit only trivial, constant boundary-layer connections), the induced right

inviscid boundary conditions are given, setting $m_0 = \rho(1)u(1)$, by

$$(4.9) \quad u(1) = u_1 (> 0),$$

in agreement with the number of incoming (hyperbolic) characteristic modes to the domain. For supersonic states $u(1) \geq \sqrt{p'(\rho(1))}$, for which all characteristic modes leave the domain, the state $(\rho(1), u(1))$ is an attractor in backward x for the boundary layer profile ODE with appropriate constant b , connecting to all states $\rho_1 \leq \rho(1)^\dagger$. Thus, there are *no imposed right boundary conditions*, in agreement with the number (zero) of incoming modes, other than the “range” condition

$$(4.10) \quad \rho_1 := m_1/u_1 \leq \rho(1)^\dagger, \quad m_1 := \rho(1)u(1)$$

on the set of allowable $u(1)$. This is always satisfied for $u(1) \leq u_1$, but for $u(1) > u_1$ imposes a strictly stronger upper bound on $\rho(1)$ than subsonicity.

Here, we have considered in our discussion of right boundary conditions only states $u(1) > 0$; the treatment of negative values would involve also “2-family” boundary layers with $m < 0$, hence additional complexity.

As all conditions are open ones, for solutions lying near a *noncharacteristic* steady solution, the operant boundary conditions are of the standard type considered in hyperbolic initial boundary value theory, hence in this local setting *the formal description of Gislsson-Serre may indeed be extended sensibly to the case of a bounded interval*, as follows also by the original observations of [GS94, Gis96, GMMZ09]. However, the range-restrictions on these boundary conditions are not consistent with standard noncharacteristic hyperbolic theory, but rather a new “obstacle” type boundary condition. Thus, both near the characteristic double layer type solutions, for which (4.8) may enter, or in the global, large-amplitude setting where (4.10) comes into play, it seems a very interesting open problem whether such a boundary condition can determine a reasonable hyperbolic flow. We emphasize for the case of double layer solutions that *even small perturbations of the background steady state lead to consideration of nonstandard boundary condition* (4.8). This might be called “transcharacteristic-type,” bridging as it does between sub- and supersonic boundary conditions, and seems one of the more interesting aspects of the investigations of this section.

More generally, the questions of global existence for large-amplitude data for the viscous problem on the interval, and convergence to a vanishing viscosity limit seem very interesting, especially in the isentropic case where the entropy-based method of *compensated compactness* has so been successful on the whole-line problem [CP10, DiP83, Ser86]. One may ask in particular whether or not boundary entropy conditions as in [DLF88] might agree with the standard and nonstandard boundary conditions derived by viscous profile considerations above.

The left boundary Riemann problem. Let us go a bit further in consideration of the transcharacteristic-type boundary conditions (4.8)–(4.10), and their role in existence/well-posedness of hyperbolic solutions near a constant characteristic solution $(\bar{\rho}, \bar{u}) \equiv (\rho_*, u_*)$,

$u_* = \sqrt{\rho_*}$. Recall that linearized and nonlinear stability hinge upon solvability of the *boundary Riemann problems* at left and right boundary of the interval, i.e., solvability on the half-line of the problem with constant initial data together with the given boundary conditions. For definiteness, let us first discuss the left boundary, renoting the constant data to the right of the boundary as (ρ_R, u_R) . Then, the solution if it exists consists of a left state $W_L = (\rho_L, u_L)$ next to the boundary and satisfying the boundary conditions, connected by a (possibly trivial, or zero-strength) 1-shock or -rarefaction to a middle state (ρ_M, u_M) , which is in turn connected by a 2-shock or -rarefaction to (ρ_R, u_R) .

We now reverse the point of view of the previous discussion and fix boundary data ρ_0, u_0 , asking in turn what states (ρ_L, u_L) , (ρ_M, u_M) and (ρ_R, u_R) it may be connected to. The set of connectible states (ρ_L, u_L) is exactly the admissible set $\mathcal{E}$ defined in [GS94]. We define the set $\mathcal{E}'$ to be the “composite” admissible set of states (ρ_M, u_M) reachable by a combination of boundary-layer and 1-wave solutions.

(i) *Subsonic case* ($\rho_0 \geq \rho_*$). A key observation is to notice that for any subsonic states (ρ_L, u_L) lying on the admissible boundary-layer curve $\rho u = m_0 = \rho_0 u_0$, the 1-characteristic has strictly negative speed leading out of the domain, hence there are no admissible 1-waves in associated boundary Riemann patterns. On the other hand, at the extreme, characteristic point $(\rho_L, u_L) = (\rho_*, u_*)$ bounding the admissible set, $u_* = \sqrt{P'(\rho_*)}$, the 1-characteristic speed is zero, and thus one may adjoin to this state a 1-rarefaction wave moving to the right into the domain, since all characteristics in the Riemann fan will have speed ≥ 0 . One cannot adjoin a 1-shock however, as its speed would be strictly less than that of the left-most 1-characteristic speed, which would be zero. Thus, the set of admissible middle states (ρ_M, u_M) consists precisely of portion of the admissible curve $\rho u = m_0$ with $\rho \geq \rho_*$, concatenated with the 1-rarefaction curve emanating from ρ_* , also pointing in the direction of decreasing ρ .

Lemma 4.2. *For $P', P'' > 0$, and $\rho_0 > \rho_*$, the composite admissibility set $\mathcal{E}'$ is a globally C^1 decreasing graph of u over $\rho \in \mathbb{R}^+$, tangent at (ρ_*, u_*) to the 1-Hugoniot curve through (ρ_*, u_*) , and transversal at every point to the 2-Hugoniot curve originating from that point.*

Proof. It is a standard fact [Smo94] that the 1-rarefaction curve through (ρ_*, u_*) is tangent to the 1-Hugoniot curve through (ρ_*, u_*) , consisting of states (ρ, u) connected to (ρ_*, u_*) by Rankine–Hugoniot conditions

$$(4.11) \quad s(\rho - \rho_*) = \rho u - \rho_* u_*, \quad s(\rho u - \rho_* u_*) = \frac{1}{2}(\rho u^2 - \rho_* u_*^2 + P(\rho) - P(\rho_*)),$$

where s is the associated shock speed. For isentropic gas dynamics under the assumptions $P', P'' > 0$, the solution may be expressed [Smo94] as global C^1 functions $u(\rho)$, $s(\rho)$, with $s(\rho_*)$ equal to the characteristic speed at ρ_* , or zero. Thus, in the vicinity of ρ_* , $s(\rho - \rho_*) = O(|\rho - \rho_*|^2)$, and so, rearranging (4.11)(i), we have $\rho u = m_0 + O(|\rho - \rho_*|^2)$, or

$$u = \frac{m_0}{\rho} + O(|\rho - \rho_*|^2),$$

giving the asserted tangency at ρ_* to the boundary admissibility curve $u = m_0/\rho$.

Transversality follows, similarly as for transversality of 1- and 2-Hugoniot curves in the study of the initial value Riemann problem [Smo94] from the fact that both are graphs over ρ , with u strictly decreasing on the first and increasing for the second. $\square$

(ii) *Supersonic case* ($\rho_0 \leq \rho_*$).

Lemma 4.3. *For $P', P'' > 0$, and $\rho_0 \leq \rho_*$, the composite admissibility set $\mathcal{E}'$ is decreasing, C^1 on $\rho \geq \rho_0^\dagger$, continuous at $\rho = \rho_0^\dagger$, and transversal at every point to the 2-Hugoniot curve originating from that point.*

Proof. Similarly as in the previous case, we have that $\mathcal{E}$ consists of the single supersonic state $(\rho_L, u_L) = (\rho_0, u_0)$ together with the subsonic states $\rho_L \geq \rho_1^\dagger$; in particular, it is disconnected. However, 1-shocks from (ρ_0, u_0) to (ρ, u) have positive speed, hence are admissible in a boundary Riemann pattern, for $\rho \leq \rho_0^\dagger$, with boundary $\rho_0^\dagger$ representing the point where shock speed is exactly zero. But, this point joins continuously to $\mathcal{E}$, giving continuity. Moving in the other direction along the 1-Hugoniot curve, with $\rho \leq \rho_0$, gives right-moving rarefaction waves, also admissible in boundary rarefaction patterns. Thus, the 1-Hugoniot curve for $\rho \leq \rho^\dagger$ joined with $\mathcal{E}$ gives the described curve $\mathcal{E}'$. One may check that the two curves are *not* tangent at $\rho = \rho_0^\dagger$, however, as the shock curve satisfies at the characteristic point $s = 0$ the derivative formula $\dot{m} = \dot{s}(\rho - \rho_0) \neq 0$, whereas the admissibility curve satisfies $\dot{m} \equiv 0$, where $m = \rho u$. $\square$

Corollary 4.4. *Assuming $P', P'' > 0$, for any right state (ρ_R, u_R) , there is a unique middle state $(\rho_M, u_M) = \Phi(\rho_R, u_R)$ lying on the composite admissibility curve $\mathcal{E}'$, hence the left boundary Riemann problem has a unique global solution. Moreover, Φ is piecewise C^1 , with discontinuous derivative along a single curve.*

Proof. From the fact that the 2-Hugoniot curve $u_2(\rho)$ through any (ρ_R, u_R) is strictly increasing for $\rho \in \mathbb{R}^+$, with limits 0 and $+\infty$ as $\rho \rightarrow 0$ and $\rho \rightarrow +\infty$, and the fact that the graph of $\mathcal{E}'$ is strictly decreasing, with limits $u \rightarrow +\infty$ and $u \rightarrow 0$ as $\rho \rightarrow 0$, $\rho \rightarrow +\infty$, we find that there is a unique intersection between the 2-Hugoniot curve and $\mathcal{E}'$. This gives existence and uniqueness; C^1 dependence then follows by the Inverse Function Theorem by C^1 regularity and transversality of the graphs of $\mathcal{E}$ and the 2-Hugoniot curve, everywhere except on the forward 2-Hugoniot curve emanating from $(\rho_0^\dagger, m_0/\rho_0^\dagger)$. $\square$

The right boundary Riemann problem. We now study the right boundary problem, with data u_1 . To match the previous discussion, we will treat ρ_1 as a fixed parameter, and define for each ρ_1 the admissible set $\mathcal{F}$ of states (ρ_R, u_R) connectible to (ρ_1, u_1) by a boundary layer. As in the previous case, we will take all velocities u to be positive. Thus, 2-waves move to the right out of the domain and cannot be part of any boundary Riemann pattern at $x = 1$. We define the composite admissible set $\mathcal{F}'$ to be the set of states (ρ_L, u_L) reachable by a (possibly trivial) leftgoing 1-shock or -rarefaction from $\mathcal{F}$.

(i) *Subsonic case* ($\rho_1 \geq \rho_*$). This case is analogous to left boundary case (ii). For the right boundary layer problem, we consider backward flow in the boundary-layer ODE, for

which subsonic equilibria are repellers, and supersonic equilibria attractors. Thus, nearby (ρ_1, u_1) , the only boundary-layer connection is the trivial one, to itself, or $(\rho_R, u_R) = (\rho_1, u_1)$. On the other hand, (ρ_1, u_1) may be connected to any supersonic state (ρ_R, u_R) with $\rho_R \leq \rho_1^\dagger$, with the limiting point $\rho_R = \rho_1^\dagger$ corresponding to a zero-speed 1-shock. Thus, $\mathcal{F}$ consists of the disconnected set of points on the graph $\rho u = m_1 := \rho_1 u_1$ lying over $\rho_R = \rho_1$ and $\rho_R \leq \rho_1^\dagger$.

There are no preceding 1-waves possible for the supersonic states $\rho_L \leq \rho_1^\dagger$, as these would be right-moving out of the domain. The subsonic state (ρ_1, u_1) however, admits a preceding 1-rarefaction from any point (ρ_L, u_L) on the 1-Hugoniot curve through (ρ_1, u_1) with $\rho_L \geq \rho_1$, and a preceding 1-shock from any point (ρ_L, u_L) on the 1-Hugoniot curve through (ρ_1, u_1) with $\rho_L \geq \rho_1^\dagger$, terminating at the point $\rho_L = \rho_1^\dagger$ corresponding to a zero-speed 1-shock.

Lemma 4.5. *For $P', P'' > 0$, $u_1 > 0$, and $\rho_1 \geq \rho_*$, the composite admissibility set $\mathcal{F}'$ is decreasing, C^1 on $\rho \geq \rho_1^\dagger$, continuous at $\rho_0^\dagger$ to the 1-Hugoniot curve through (ρ_0, u_0) , and transversal at every point to the 2-Hugoniot curve originating from that point.*

Proof. Essentially identical to the proof of Lemma 4.3. $\square$

(ii) *Supersonic case* ($\rho_1 \leq \rho_*$). In this case, which is analogous to left boundary case (i), (ρ_1, u_1) may be connected by boundary layer to any supersonic state, i.e., to any state (ρ_R, u_R) on the curve $\rho u \equiv m_1$ with $\rho_R \leq \rho_*$, with the limiting point $\rho_R = \rho_*$ corresponding to a characteristic boundary layer. The set $\mathcal{F}$ thus consists of all such supersonic states. From strictly supersonic states, no preceding 1-wave is possible, since it would be right-moving out of the domain. However, the characteristic boundary point $\rho_R = \rho_*$ may be preceded by a 1-rarefaction from any point $\rho_L \geq \rho_*$. The composite admissible set $\mathcal{F}'$ thus consists of the concatenation of the boundary-layer curve up to $\rho = \rho_*$ with the 1-rarefaction curve emanating from $(\rho_*, m_1/\rho_*)$.

Lemma 4.6. *For $P', P'' > 0$, $u_1 > 0$, and $\rho_1 \leq \rho_*$, the composite admissibility set $\mathcal{F}'$ is decreasing, globally C^1 , and tangent at $\rho = \rho_*$ to the 1-Hugoniot curve through $(\rho_*, m_1/\rho_*)$.*

Proof. Essentially identical to the proof of Lemma 4.2. $\square$

Corollary 4.7. *Assuming $P', P'' > 0$, $u_1 > 0$, any left state (ρ_L, u_L) , $u_L > 0$, lies on lying on the composite admissibility curve $\mathcal{F}'$ for a unique boundary value*

$$(\rho_1, u_1) = (\phi(\rho_L, u_L, u_1), u_1).$$

Moreover, Φ is piecewise C^1 with discontinuous derivative along a 2D surface in (ρ_L, u_L, u_1) .

Proof. By Galilean invariance, the 1-Hugoniot curves through (ρ_1, u_1) are translates of each other in u and cover the ρ - u plane in one-to-one fashion as u_1 is varied. Likewise, the curves $\rho u = m_1$ cover the plane in one-to-one fashion as m_1 is varied. The case (i) cuts these families along the line $\rho = \rho_1^\dagger$ and glues left and right halves together; moreover, the cutoff curve may be seen to intersect curves in each family precisely once. At the

limiting case $\rho_1 = \rho_*$, this cutoff is precisely at ρ_* . Case (ii) is a similar gluing of the same two families of curves, but this time cut at ρ_* , again with a single intersection of cutting curve with individual curves in each family. Thus, the two halves together cover the plane in one-to-one fashion, giving a unique solution of the right boundary Riemann solution. By the Implicit Function Theorem, moreover, this is C^1 except where (ρ_1, u_1) is subsonic, with $\rho_1 u_1 = m_1 = \rho_L u_L$ and $\rho_1^\dagger = \rho_L$, tracing out a single curve with one boundary of discontinuity in the (ρ_L, u_L) plane, with vertex at the characteristic point $\rho_L u_L / u_1 = \rho_*$. Taking the union over u_1 , we are done. $\square$

Remark 4.8. *The constructions of Lemma 4.2-4.6 are reminiscent of composite wave constructions in initial value Riemann solutions for nonconvex conservation laws [Daf85, Liu81].*

Well-posedness near steady solutions. Reviewing Corollaries 4.4 and 4.7 we see that small perturbations of steady hyperbolic solutions lead to boundary Riemann problems in the C^1 part of the solution operator. In particular, in the new case of the double characteristic boundary-layer solution, we have (ρ_0, u_0) subsonic and (ρ_1, u_1) supersonic, with middle state $(\hat{\rho}, \hat{u}) \equiv (\rho_*, u_*)$ characteristic, where $\rho u \equiv m_0 = \rho_0 u_0$ for all states. Thus, small perturbations of the interior solution lead to boundary Riemann problems in the C^1 cases described in Lemmas 4.2 and 4.6.

This yields well-posedness in the Sobolev sense of Kreiss and Majda, as considered in [GS94] of small perturbations of steady hyperbolic solutions of the inflow/outflow problem. Behavior for global data remains a very interesting open problem, as it is also in more standard cases. The effect of discontinuous derivatives in the boundary Riemann solver for small data solutions is another very interesting open question.

Entropy boundary conditions. In [DLF88] there is derived for general conservation laws (1.1) the (left) entropy boundary inequality

$$(4.12) \quad q(U(0)) - q(U_0) \leq d\eta(U_0)(U(0) - U_0),$$

for any entropy/entropy flux pair η (not necessarily convex)/ q such that $\Re(d^2\eta(U)B(U)) \geq 0$, also called “ B -convex.” A symmetric inequality holds for the right boundary. As shown in [GS94, Prop. 2], the Gisclon-Serre type boundary condition obtained from viscous boundary-layer criteria necessarily satisfies (4.12). In [DLF88, Prop. 1.1], it is shown in the linear case that conditions (4.12) agree with a formulation in terms of Riemann problems that enforces unique solutions, hence the two concepts must agree. However, in the general nonlinear case they clearly differ, as there does not necessarily exist any B -convex entropy. For the case of isentropic gas dynamics it was conjectured in [DLF88] that they agree, but to our knowledge this question has up to now neither been proved nor disproved.

Equivalence of the two types of boundary condition would be extremely interesting, as it would imply that *any* weak vanishing viscosity solution would satisfy Gisclon-Serre type boundary conditions, whereas in [GS94, Prop. 1] it is shown that smooth (noncharacteristic) Gisclon-Serre type solutions are vanishing viscosity limits of *some* sequence of solutions.

4.2 The standing shock limit

A simple case in which the zero-viscosity limit can be completely carried out is that of the “standing-shock limit” generalizing the study of [Zum10] in the case of the half-line. This consists of the study of a stationary viscous n -shock $\hat{U}^\varepsilon(x) = \bar{U}(\frac{1}{\varepsilon}(x - \frac{1}{2}))$ of (4.1), solving (4.2) for all $\varepsilon > 0$, with respect to its “own” boundary conditions, i.e.

$$U_0 = \hat{U}^\varepsilon(0), \quad U_1^{II} = \hat{U}^\varepsilon(1).$$

We consider this for the general class of system (1.1),(1.2) under assumptions **(H0)**–**(H3)** plus the additional assumption used in [Zum10]:

(H4) the eigenvalues of $df(U_0)$ and $df(U_1)$ are nonzero.¹

Converting by $x \rightarrow \frac{x}{\varepsilon}$ to the large-interval limit and following the arguments of [Zum10] word for word, we find that, away from $\lambda = 0$, the spectra of the linearized operator about $\hat{U}^X(x) := \hat{U}^\varepsilon(\varepsilon x)$ on $\Re \lambda \geq 0$ approaches, as $\varepsilon = \frac{1}{X} \rightarrow 0^+$, the direct sum of the spectra of the viscous shock $\bar{U}$ as a solution on the whole line plus the spectra of the constant boundary layers on the half-lines $(0, +\infty)$ and $(-\infty, 1)$ determined by the values of $\hat{U}^\varepsilon$ at 0 and 1 with the boundary conditions for the steady problem at $x = 0$ and $x = 1$. As the latter constant layers have been shown to be spectrally stable [GMWZ05], this implies that the spectra of $\hat{U}^X$ converges away from $\lambda = 0$ to that of $\bar{U}$ as $X \rightarrow \infty$. Rescaling, we find that, outside $B(0, c\varepsilon^{-1})$, any $c > 0$, the spectra of $\hat{U}^\varepsilon$ are well-approximately by ε^{-1} times the spectra of $\bar{U}$. We record this observation as the following proposition.

Proposition 4.9 (Spectral decomposition). *For viscous n -shock solutions $\bar{U}$ of systems (1.1) satisfying **(H0)**–**(H4)**, the corresponding standing-shock family $\bar{U}^\varepsilon$ contains no spectra $\Re \lambda \geq 0$ outside a ball $B(0, c\varepsilon^{-1})$ for $\varepsilon > 0$ sufficiently small, for any choice of $c > 0$, if and only if $\bar{U}$ is spectrally stable, i.e., has no spectra $\Re \lambda \geq 0$ with $\lambda \neq 0$. In particular, if $\bar{U}$ is spectrally unstable, then $\hat{U}^\varepsilon$ is spectrally unstable for ε sufficiently small.*

Proposition 4.9 gives no information about the corresponding stability index and uniqueness or nonuniqueness. However, this is provided definitively by the following result.

Proposition 4.10 (Nonvanishing of the stability index). *For viscous n -shock solutions $\bar{U}$ of systems (1.1) satisfying **(H0)**–**(H4)**, the Evans function D^ε associated with the corresponding standing-shock family $\bar{U}^\varepsilon$ satisfies $D^\varepsilon(0) \neq 0$ for $\varepsilon > 0$ sufficiently small.*

Proof. We only sketch the proof, which belongs more to the circle of ideas in [Zum10] than those of the present paper. We first write the eigenvalue system in “flux” variables (u_{II}, F) as

$$(4.13) \quad \begin{aligned} U'_{II} &= B_{22}(U)^{-1}(F_{II} + A_{11}U_I), \\ F' &= \lambda A^0 U, \end{aligned}$$

¹By **(H3)** the eigenvalues of $df(U)$ are real and semi-simple since $df(U)$ is symmetric for the inner product associated to $S(U)$.

where $F := B(\hat{U})U' - AU$ and $U_I = A_{11}^{-1}(F_I + A_{12}U_{II})$. This yields for $\lambda = 0$ in the second equation the simple dynamics $F \equiv \text{constant}$.

Next, we observe that the Evans function may be written equivalently as

$$D^\varepsilon(\lambda) = \det \begin{pmatrix} U_{II}^1 & \cdots & U_{II}^r & 0 \\ F^1 & \cdots & F^r & I_n \end{pmatrix} \Big|_{x=1},$$

where $\begin{pmatrix} U_{II}^j \\ F^j \end{pmatrix}$, $j = 1, \dots, r$ denote the solutions of (4.13) with initial conditions

$$(4.14) \quad \begin{pmatrix} U_{II}^1 & \cdots & U_{II}^r \\ F^1 & \cdots & F^r \end{pmatrix} (0) = \begin{pmatrix} 0 \\ B(0) \begin{pmatrix} 0 \\ I_r \end{pmatrix} \end{pmatrix}$$

at $x = 0$. In turn, we may view this as a Wronskian

$$(4.15) \quad D^\varepsilon(\lambda) = \det \begin{pmatrix} U_{II}^1 & \cdots & U_{II}^r & U_{II}^{r+1} & \cdots & U_{II}^{n+r} \\ F^1 & \cdots & F^r & F^{r+1} & \cdots & F^{n+r} \end{pmatrix} \Big|_{x=1},$$

where $\begin{pmatrix} U_{II}^j \\ F^j \end{pmatrix}$, $j = r+1, \dots, n+r$ denote the solutions of (4.13) with initial conditions

$$(4.16) \quad \begin{pmatrix} U_{II}^{r+1} & \cdots & U_{II}^{r+n} \\ F^{r+1} & \cdots & F^{r+n} \end{pmatrix} (1) = \begin{pmatrix} 0 \\ I_n \end{pmatrix}$$

at $x = 1$.

By Abel's theorem, vanishing or nonvanishing of the Wronskian (4.15) at $x = 1$ is determined by vanishing or nonvanishing at any $x \in [0, 1]$. By the analysis of [Zum10], we find that, at $x = c$ for any $c > 0$ sufficiently small, the solutions $\begin{pmatrix} U_{II}^j \\ F^j \end{pmatrix}$, $j = 1, \dots, r$ originating from $x = 0$ converge exponentially in $X := 1/\varepsilon$ to the limiting subspace of solutions of (4.13) on the whole line decaying at $x = -\infty$, which may be identified by the property $F \equiv 0$, hence also $\det(U_{II}^1, \dots, U_{II}^r) \neq 0$. Recalling by the simple dynamics for $\lambda = 0$ that

$$(F^{r+1}, \dots, F^{r+n}) \equiv \begin{pmatrix} B(0) \begin{pmatrix} 0 \\ I_r \end{pmatrix} \end{pmatrix},$$

we find that the Wronskian at $x = c$ converges exponentially in $X = \varepsilon^{-1}$ to

$$\det(U_{II}^1, \dots, U_{II}^r) \Big|_{x=c} \neq 0,$$

hence $D^\varepsilon(0) \neq 0$ for $\varepsilon > 0$ sufficiently small. For further details, see [Zum10]. $\square$

Remark 4.11. *The result of Proposition 4.10, though proved by similar techniques, stands in striking contrast to the results of [Zum10, SZ01] in the half-line case, where the stability index was seen to change sign as parameters were varied for (full) polytropic gas dynamics, despite stability of the underlying whole-line shock profile. The standing-shock construction can nonetheless be useful in seeking instability/bifurcation, however, through bifurcation from unstable to stable background shock; this strategy is used successfully, e.g., in [BMZ21].*

4.3 Large-viscosity/small interval asymptotics

It seems interesting to ask also what are the asymptotics of solutions as viscosity goes to infinity instead of zero, or, equivalently, interval length goes to zero with viscosity fixed. Before studying the general case (under reasonable assumptions) we provides two examples that illustrate what can happen for such an asymptotic.

4.3.1 The scalar case

We consider here the scalar case as in Section 3. For a $\mathcal{C}^1$ function f , $\nu \geq 1$ and $u_0, u_1 \in \mathbb{R}$, we consider the unique solution u^ν (as proved in Proposition 3.1) of

$$(4.17) \quad \begin{cases} \nu \hat{u}_{xx}^\nu = (f(\hat{u}^\nu))_x, & 0 < x < 1, \\ \hat{u}^\nu(0) = u_0, \hat{u}^\nu(1) = u_1. \end{cases}$$

The following proposition provides the behavior of $\hat{u}^\nu$ as $\nu \rightarrow +\infty$.

Proposition 4.12. *Let $\nu \geq 1$ and $u_0, u_1 \in \mathbb{R}$ and assume that f is $\mathcal{C}^1(\mathbb{R})$. Consider $\hat{u}^\nu$ the unique solution of (4.17). There exists a constant $C > 0$ depending only on f , u_0 and u_1 such that for any $x \in [0, 1]$,*

$$|\hat{u}^\nu(x) - ((u_1 - u_0)x + u_0)| \leq \frac{C}{\nu}.$$

Proof. Integrating the equation from 0 to $x \in [0, 1]$, there exists a constant $C_1 > 0$ depending only on f , u_0 and u_1 so that

$$|\hat{u}_x^\nu(x) - u_x^\nu(0)| \leq \frac{C_1}{\nu}.$$

Since $\min(u_0, u_1) \leq u^\nu \leq \max(u_0, u_1)$, $\hat{u}_x^\nu$ is therefore necessarily uniformly bounded with respect to $\nu \geq 1$. Thus $\nu \hat{u}_{xx}^\nu$ is uniformly bounded with respect to $\nu \geq 1$. Integrating two times, we obtain that the quantity $\nu(u_1 - u_0 - \hat{u}_x^\nu(0))$ is uniformly bounded with respect to $\nu \geq 1$. The desired estimate follows. $\square$

4.3.2 Full gas dynamics

For full gas dynamics [BMZ21, Eq. (2.6)], (2.3) becomes

$$(4.18) \quad \frac{\alpha}{u_0} u' = c_1 + u + \Gamma \frac{e}{u}, \quad \frac{\nu}{u_0} e' = c_2 - c_1 u - \frac{1}{2} u^2 + e,$$

where $u > 0$ is fluid velocity, $e > 0$ specific internal energy, and

$$(4.19) \quad c_1 = \frac{\alpha}{u_0} u'(0) - u_0 - \Gamma \frac{e_0}{u_0}, \quad c_2 = \frac{\nu}{u_0} e'(0) + \alpha u'(0) - e_0 - \frac{1}{2} u_0^2 - \Gamma e_0.$$

(Here, we have taken without loss of generality $\rho_0, u_0 = 1$, so that by the steady mass conservation equation, $\rho u \equiv 1$, where ρ is fluid density [BMZ21].)

Formally taking α, ν to infinity, and dropping $O(1)$ terms, we obtain limiting equations $\frac{\alpha}{u_0}u' = c_1$, $\frac{\nu}{u_0}e' = c_2 - c_1u$ with $c_1 = \frac{\alpha}{u_0}u'(0)$, $c_2 = \frac{\nu}{u_0}e'(0)$, or

$$(4.20) \quad \bar{u}' = u'(0), \quad \frac{1}{u_0}\bar{e}' = \bar{e}'(0) - \frac{\alpha}{\nu}\bar{u}'(0)\bar{u},$$

giving exact solution

$$(4.21) \quad \bar{u}(x) = u_0 + x(u_1 - u_0), \quad \bar{e}(x) = x\left(e_1 - \frac{\alpha}{\nu}\frac{u_1^2 - u_0^2}{2}\right) + \frac{\alpha}{\nu}(u_1 - u_0)(u_0x + (u_1 - u_0)x^2/2).$$

For $\varepsilon > 0$, set $E_\varepsilon = \{(u, e) \in \mathbb{R}^2, \varepsilon < u, e < \frac{1}{\varepsilon}\}$.

Proposition 4.13. *For any fixed $\varepsilon > 0$ and profile (4.21) contained in E_ε , holding $r = \nu\alpha > 0$ fixed, and taking α, ν to infinity, there is for sufficiently large (α, ν) a unique steady profile of (4.18) lying in E_ε , converging to the formal limit (4.21) in $H^1[0, 1]$ at rate $O(\alpha^{-1})$.*

Proof. Put back into second-order form, (4.18) becomes (cf. [BMZ21, Eq. (2.5)])

$$(4.22) \quad \left(\frac{\alpha}{u_0}u'\right)' = (u + \Gamma\frac{e}{u})', \quad \left(\frac{\nu}{u_0}e' + \frac{\alpha}{u_0}uu'\right)' = (\Gamma e + e + \frac{1}{2}u^2)',$$

Writing $u = \bar{u} + w$, $e = \bar{e} + z$, we have, substituting in (4.22), subtracting the corresponding equation for $\bar{u}$, dividing by α , the elliptic perturbation equations

$$(4.23) \quad \left(\frac{1}{u_0}w'\right)' = (O(\alpha^{-1}))', \quad \left(\frac{r}{\alpha u_0}e' + \frac{1}{u_0}(\bar{u}w' + w\bar{u}' + ww')\right)' = +(O(\alpha^{-1}))',$$

with homogeneous data $(w, z)(0) = (0, 0)$, $(w, z)(1) = (0, 0)$, where estimate $O(\alpha^{-1})$ (by Sobolev embedding) remains valid so long as $\|w\|_{H^1[0,1]}$ is sufficiently small.

With this structure, it is straightforward to carry out a contraction mapping argument for w in a sufficiently small ball B in $H^1[0, 1]$, considering the right-hand side as input and the solution of the block-triangular operator on the left-hand side as image, and showing by energy estimates that the resulting operator is contractive on B with contraction constant $O(\alpha^{-1}) < 1$, giving existence and uniqueness by the Banach fixed point theorem of a solution in B , i.e., small in $H^1[0, 1]$. Thus, the righthand side of (4.23) is $O(\alpha^{-1})$, and, solving, we have $\|(w, z)\|_{H^1[0,1]} = O(\alpha^{-1})$ as well. Recalling that (w, z) by definition is the difference between exact and limiting solutions, we are done. $\square$

Remark 4.14. *The large-viscosity analysis for isentropic gas dynamics is similar but much simpler, yielding $\bar{u}$ as the line interpolating between $u(0)$ and $u(1)$. This is to be compared with the rich small-viscosity behavior depicted in Figure 1.*

4.3.3 General case

Now, let us consider the general case, under assumption (2.15) and, possibly after a coordinate change, B_{22} symmetric positive definite: this is possible for example for systems with convex entropy [KS88]. Adding a variable viscosity ν in (1.1) gives

$$\partial_t f^0(U) + f(U)_x = \nu (B(U)U_x)_x,$$

leading to the steady problem

$$(4.24) \quad \nu B_{22}(U)U'_{II} = f_{II}(U) - f_{II}(U_0) + \nu \tilde{C}_2, \quad U_{II}(0) = U_{0II}, \quad f_I(U) = f_I(U_0),$$

where $\tilde{C}_2 = B_{22}(U_0)C_2 = B_{22}U'_{II}(0)$, and functions of U are viewed as functions on U_{II} alone, through the dependence of U_I on U_{II} imposed by relation (2.15).

As in the previous case, we expect that solutions will converge to the solution of the formal limiting equations obtained by multiplying by $\nu^{-1}B_{22}(\bar{U}_{II})^{-1}$ and taking $\nu \rightarrow \infty$, or

$$(4.25) \quad \bar{U}'_{II} = B_{22}(\bar{U}_{II})^{-1}\tilde{C}_2, \quad \bar{U}_{II}(0) = U_{0II}, \quad f_I(\bar{U}) = f_I(U_0).$$

A first question, partially answered in the next lemma, is whether (4.25) admits a solution.

Lemma 4.15. *Let $0 < \beta_0 < B_{22}^{-1} < \beta_1$ on a subset $\tilde{\mathcal{U}}$ of the domain of definition $\mathcal{U}$ of (4.24), and set $\theta = \cos^{-1}(\beta_0/\beta_1)$. If the truncated cone $\mathbb{T}$ of angle θ based at U_{0II} and centered about the segment from U_{0II} to $U_{0II} + (\beta_1/\beta_0)(U_{1II} - U_{0II})$ is contained in $\tilde{\mathcal{U}}$, then (i) any steady solution $\hat{U}$ valued in $\tilde{\mathcal{U}}$ of the formal limiting equation (4.25) lies also in $\mathbb{T}$, and (ii) there is at least one steady solution $\hat{U}$ lying in $\mathbb{T}$, hence valued in $\tilde{\mathcal{U}}$.*

Proof. By $\beta_0 < B_{22}^{-1} < \beta_1$, for any vector V the angle θ between V and $B_{22}^{-1}V$ satisfies

$$\cos \theta = \frac{\langle V, B_{22}^{-1}V \rangle}{|V||B_{22}^{-1}V|} \geq \frac{\beta_0}{\beta_1},$$

with length of $B_{22}^{-1}V$ lying between $\beta_0|V|$ and $\beta_1|V|$.

Likewise, so long as U_{II} remains in $\tilde{\mathcal{U}}$, for $0 \leq x \leq 1$ the vector

$$U_{II}(x) - U_{II}(0) = \left(\int_0^x B_{22}^{-1}(U_{II}(y))dy \right) \tilde{C}_2 = B_{ave}(x\tilde{C}_2),$$

by the corresponding bounds $\beta_0 < B_{ave} < \beta_1$, lies within the same angle θ of $\tilde{C}_2$, with length between $x\beta_0|\tilde{C}_2|$ and $x\beta_1|\tilde{C}_2|$. If U_{II} is a solution of (4.24), i.e., $U_{II}(0) = U_{0II}$ and $U_{II}(1) = U_{122}$, then this implies that $\tilde{C}_2$ is within angle θ of $U_{1II} - U_{0II}$, and thus $U_{II}(x)$, lying within angle θ of $\tilde{C}_2$, lies in the truncated cone $\mathbb{T}$.

On the other hand, for $\tilde{C}_2$ lying within the narrower truncated cone $\mathbb{T}_2$ of angle θ around $V = U_{1II} - U_{0II}$, and length between $|V|/\beta_1$ and $|V|/\beta_2$, we have by the same estimates that U_{II} stays within the interior of $\mathbb{T}$ for $0 \leq x \leq 1$, hence $\Psi(\tilde{C}_2) = U_{II}(1) - U_{0II}$ is well-defined. Varying C_2 around the boundary of $\mathbb{T}_2$, we find by the same angle estimate that the degree of Ψ about 0 is +1, giving existence of at least one steady profile in $\mathbb{T}$. $\square$

Remark 4.16. By a constant change of coordinates, we may take

$$B_{22}(U_{II}) \rightarrow \tilde{B}_{22}(U_{II}) = B_{22}(U_{0II})^{-1/2} B_{22}(U_{II}) B_{22}(U_{0II})^{-1/2},$$

in particular $\tilde{B}_{22}(U_{0II}) = \text{Id}$, to improve the condition number β_1/β_0 . For $B_{22} \equiv \text{constant}$, as for gas dynamics, this yields $\tilde{B}_{22} \equiv \text{Id}$, giving an exact solution $U_{II}(1) = \tilde{B}_{22}^{-1} \tilde{C}_2$. In general, we do not see that there is necessarily a solution of (4.24) lying in $\tilde{\mathcal{U}}$, even “nice” domains arising in applications, nor that solutions of (4.24) must necessarily be unique.

Mimicking our treatment of the gas dynamical case, we may write (4.24) in elliptic form

$$(4.26) \quad (\nu B_{22}(U) U'_{II})' = (f_{II}(U))', \quad U_{II}(0) = U_{0II}, \quad f_I(U) = f_I(U_0),$$

then, defining $W = \hat{U}_{II} - \bar{U}_{II}$ to be the difference between exact and limiting solutions, rewrite as an elliptic perturbation system

$$(4.27) \quad (B_{22}(\bar{U})W' + (dB_{22}(\bar{U})W)\bar{U}'_{II})' = ((dB_{22}(\bar{U})W)W')' + (O(\nu^{-1}))',$$

with Dirichlet boundary conditions $W(0) = W(1) = 0$. If, as in the gas-dynamical case, the operator on the lefthand side is uniformly elliptic, then obtain an existence/convergence result as in Proposition 4.13. by the same sort of contraction mapping argument used there. However, we do not see in general why this should be true, *except for small data* $|\bar{U}'| \ll 1$.

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