

On the variational principle.

Ivar Ekeland,

CEREMADE, Université Paris-Dauphine

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Charles University

1: Variational principles: old and new

The classical variational principle

Definition

Let X be a Banach space. We shall say that $f : X \rightarrow \mathbb{R}$ is *Gâteaux-differentiable* at x if there exists a continuous linear map $Df(x) : X \rightarrow X^*$ such that

$$\forall \xi \in X, \quad \lim_{t \rightarrow 0} \frac{1}{t} [f(x + t\xi) - f(x)] = \langle Df(x), \xi \rangle$$

In other words, the restriction of f to every line is differentiable (for instance, partial derivatives exist). If the map $x \rightarrow Df(x)$ from X to X^* is norm-continuous, f is called C^1 .

Theorem (Classical variational principle)

If f attains its minimum on X at a point $\bar{x}$, then $Df(\bar{x}) = 0$. If f is C^2 , then the Hessian $D^2f(x)$ is non-negative

Any point where $Df(\bar{x}) = 0$ is called a *critical point*.

Maximum, minimum or critical point ?

- Hero of Alexandria (3d century BCE) : light takes the shortest path from A to B. Deduces the laws of reflection
- Fermat (17th century): light takes the quickest path from A to B. Deduces the law of refraction
- Maupertuis (18th century): every mechanical system, when going from one configuration to another, minimizes the action. This is the **least action principle**.

In fact, nature is not interested in minimizing or maximizing. It is interested in critical points. The laws of physics are not

$$f(x) = \min$$

They are of the form

$$f'(x) = 0$$

The mathematical difficulty is that x is not a point, but a **path**, i.e. a map from R to R^n (ODE) or a map from R^p to R^n (PDE). The path space will be infinite-dimensional, and cannot be compact.

The billiard:

Theorem (Birkhoff)

Any convex billiard table has two diameters

The large diameter is obtained by maximizing the distance between two points on the boundary. But where is the small diameter ?

The extended variational principle (EVT)

Theorem (Ekeland, 1972)

Let (X, d) be a complete metric space, and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous map, bounded from below:

$$\{(x, a) \mid a \geq f(x)\} \text{ is closed in } X \times \mathbb{R} \text{ and } f(x) \geq 0$$

Suppose $f(x_0) < \infty$. Then for every $r > 0$, there exists some $\bar{x}$ such that:

$$\begin{aligned} f(\bar{x}) &\leq f(x_0) \\ d(\bar{x}, x_0) &\leq r \\ f(x) &\geq f(\bar{x}) - \frac{f(x_0)}{r} d(x, \bar{x}) \quad \forall x \end{aligned}$$

So (a) $\bar{x}$ is better than x_0 (b) $\bar{x}$ can be chosen as close to x_0 as one wishes
(c) $\bar{x}$ satisfies a cone condition

From now on, X will be a Banach space with $d(x, y) = \|x - y\|$

Definition

$f : X \rightarrow \mathbb{R}$ is **ε -supported** at x if there exists some $\eta > 0$ and some $x^* \in X^*$ such that:

$$\|x - y\| \leq \eta \implies f(y) - f(x) - \langle x^*, y - x \rangle \geq -\varepsilon \|y - x\|$$

Theorem

*Suppose there is a C^1 function $\varphi : X \rightarrow \mathbb{R}$ such $\varphi(0) = 1$, $\varphi(x) = 0$ for $\|x\| \geq 1$ and $\varphi \geq 0$. Then, for every lower semi-continuous function $f : X \rightarrow \mathbb{R}$ and every $\varepsilon > 0$, the set of points x where f is **ε -supported** is **dense** in X .*

Differentiability of convex functions.

Recall that, in a complete metric space, a countable intersection of open dense sets is dense (Baire).

Corollary

*For every **convex** function f on X , there is a subset $\Delta \subset X$, which is a countable intersection of open dense subsets, such that F is Fréchet-differentiable at every point $x \in \Delta$:*

$$\|x - y\| \leq \eta \implies \|f(y) - f(x) - \langle x^*, y - x \rangle\| \leq \varepsilon \|y - x\|$$

2: Optimization

First-order version of EVP

Theorem

Let X be a Banach space, and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous map, non-negative and G -differentiable. Then for every $x_0 \in X$ and $r > 0$, there exists some $\bar{x}$ such that:

$$\begin{aligned} f(\bar{x}) &\leq f(x_0), \\ \|\bar{x} - x_0\| &\leq r \\ \|Df(\bar{x})\|^* &\leq \frac{f(x_0)}{r} \end{aligned}$$

Corollary

There is a sequence x_n such that:

$$\begin{aligned} f(x_n) &\rightarrow \inf f \\ Df(x_n) &\rightarrow 0 \end{aligned}$$

For simplicity, take $x_0 = 0$ and $\inf f = 0$. Apply EVP to $x = \bar{x} + tu$ and let $u \rightarrow 0$. We get:

$$f(\bar{x} + tu) \geq f(\bar{x}) - \frac{f(0)}{r} t \|u\| \quad \forall (t, u)$$

$$\lim_{t \rightarrow +0} \frac{1}{t} (f(\bar{x} + tu) - f(\bar{x})) \geq -\frac{f(0)}{r} \|u\| \quad \forall u$$

$$\langle Df(x), u \rangle \geq -\frac{f(0)}{r} \|u\| \quad \forall u, \text{ or } \|Df(x)\|^* \leq \frac{f(0)}{r}$$

Existence problems

The compact case

Theorem

Suppose f is a lower semi-continuous function on a compact space. Then it attains its minimum

Proof.

There is a sequence x_n such that $f(x_n) \rightarrow \inf f$ and $Df(x_n) \rightarrow 0$. By compactness, it has a subsequence which converges to some $\bar{x}$, and by semi-continuity, $f(\bar{x}) \leq \liminf f(x_n) = \inf f$ □

Existence problems

The complete case

Definition

We shall say that f satisfies the **Palais-Smale condition**(PS) if every sequence x_n such that $f(x_n)$ converges and $Df(x_n) \rightarrow 0$ has a convergent subsequence

Theorem (Palais and Smale, 1964)

Any lower semi-continuous function on a Banach space, which is G -differentiable, bounded from below, and satisfies PS attains its minimum

Proof.

By EVP, there is a sequence x_n such that $f(x_n) \rightarrow \inf f$ and $Df(x_n) \rightarrow 0$. By PS, it has a subsequence which converges to some $\bar{x}$, and by semi-continuity, $f(\bar{x}) \leq \liminf f(x_n) = \inf f$ □

The EVP allows us to consider **more restricted classes of minimizing sequences**

Theorem (Borwein-Preiss, 1987)

Let X be a Hilbert space, and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a C^2 function, bounded from below. Then for every minimizing sequence x_n of f in X , there exists a sequence y_n of f such that:

$$f(y_n) \rightarrow \inf f$$

$$\|x_n - y_n\| \rightarrow 0$$

$$Df(y_n) \rightarrow 0$$

$$\liminf_n (D^2 f(y_n) u, u) \geq 0 \quad \forall u \in X$$

So we can consider **even more restricted classes of minimizing sequences**, those along which $Df(x_n) \rightarrow 0$ and $\liminf Df(x_n) \geq 0$.

3: Critical point theory

The mountain-pass theorem

Existence

Theorem (Ambrosetti and Rabinowitz 1973)

Let f be a continuous G -differentiable function on X . Assume that $f' : X \rightarrow X^*$ is norm-to-weak* continuous and satisfies PS. Take two points x_0 and x_1 in X , and define:

$$\Gamma := \{c \in C^0([0, 1]; X) \mid c(0) = x_0, c(1) = x_1\}$$

$$\gamma := \inf_{c \in \Gamma} \max_{0 \leq t \leq 1} f(c(t))$$

Assume that $\gamma \geq \max(f(x_0), f(x_1))$. Then there exists some point $\bar{x}$ with

$$f(\bar{x}) = \gamma$$

$$Df(\bar{x}) = 0$$

The mountain-pass theorem

Characterization

Theorem (Hofer 1983, Ghoussoub-Preiss 1989)

The critical point $\bar{x}$ in the preceding theorem must satisfy one of the following:

- *either there is a sequence x_n of (weak) local maxima such that $x_n \rightarrow \bar{x}$*
- *or $\bar{x}$ has **mountain-pass type**, namely, there exists a neighbourhood $\mathcal{U}$ of $\bar{x}$ such that $\mathcal{U} \cap \{f < \gamma\}$ is neither empty nor connected.*

Consider the set of paths Γ and endow it with the uniform distance. It is a complete metric space. Consider the function $\varphi : \Gamma \rightarrow \mathbb{R}$ defined by:

$$\varphi(c) := \max_{0 \leq t \leq 1} f(c(t))$$

It is bounded from below by γ , and lower semi-continuous. We can apply EVP, we get a path c_ε , take t_ε where $f(c_\varepsilon(t))$ attains its maximum, and set $x_\varepsilon = c_\varepsilon(t_\varepsilon)$. We show that $\|Df(x_\varepsilon)\|^* \leq \varepsilon$, we let $\varepsilon \rightarrow 0$ and we apply PS.

4: Inverse function theorems

A non-smooth inverse function theorem

Theorem (Ekeland, 2012)

Let X and Y be Banach spaces. Let $F : X \rightarrow Y$ be continuous and G -differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded in a neighbourhood of 0:

$$\begin{aligned}\forall v \in Y, \quad DF(x) L(x) v &= v \\ \sup \{ \|L(x)\| \mid \|x\| \leq R \} &< m\end{aligned}$$

Then, for every $\bar{y}$ such that

$$\|\bar{y}\| \leq \frac{R}{m}$$

there is some $\bar{x}$ such that:

$$\begin{aligned}\|\bar{x}\| &\leq m \|\bar{y}\| \\ F(\bar{x}) &= \bar{y}\end{aligned}$$

The standard inverse function theorem requires F to be C^1 and does not

Consider the function $f : X \rightarrow \mathbb{R}$ defined by:

$$f(x) = \|F(x) - \bar{y}\|$$

It is continuous and bounded from below, so that we can apply EVP with $r = m \|\bar{y}\|$. We can find $\bar{x}$ with:

$$f(\bar{x}) \leq f(0) = \|\bar{y}\|$$

$$\|\bar{x}\| \leq m \|\bar{y}\| \leq R$$

$$\forall x, \quad f(x) \geq f(\bar{x}) - \frac{f(0)}{m \|\bar{y}\|} \|x - \bar{x}\| = f(\bar{x}) - \frac{1}{m} \|x - \bar{x}\|$$

I claim $F(\bar{x}) = \bar{y}$.

Proof (ct'd)

Assume $F(\bar{x}) \neq \bar{y}$. The last equation can be rewritten:

$$\forall t \geq 0, \forall u \in X, \quad \frac{f(\bar{x} + tu) - f(\bar{x})}{t} \geq -\frac{1}{m} \|u\|$$

Simplify matters by assuming X is Hilbert. Then:

$$\left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}\|}, DF(\bar{x})u \right) = \langle Df(\bar{x}), u \rangle \geq -\frac{1}{m} \|u\|$$

We now take $u = -L(\bar{x})(F(\bar{x}) - \bar{y})$, so that $DF(\bar{x})u = -(F(\bar{x}) - \bar{y})$.

We get a contradiction:

$$\|F(\bar{x}) - \bar{y}\| \leq \frac{1}{m} L(\bar{x}) \|F(\bar{x}) - \bar{y}\| < \|F(\bar{x}) - \bar{y}\|$$

Global inverse function theorems

Taking $R = \infty$ gives:

Theorem

Let X and Y be Banach spaces. Let $F : X \rightarrow Y$ be continuous and Gâteaux-differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded on X

$$\forall v \in Y, \quad DF(x) L(x) v = v$$

$$\sup_x \|L(x)\| < m$$

Then, for every $\bar{y}$ there is some $\bar{x}$ such that:

$$\|\bar{x}\| \leq m \|\bar{y}\|$$

$$F(\bar{x}) = \bar{y}$$

The C^1 version is due to Hadamard.

I.Ekeland, "*Nonconvex minimization problems*", Bull. AMS 1 (New Series) (1979) p. 443-47

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N. Ghoussoub, "*Duality and perturbations methods in critical point theory*", Cambridge Tracts in Mathematics 107, 1993

All papers can be found on my website:

<http://www.ceremade.dauphine.fr/~ekeland/>

THANK YOU VERY MUCH